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DETERMINATION OF THE OPTIMUM COMPOSITION AND PROGRAM OF TRAJECTORY MEASUREMENTS

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N. N. Kozlov

ABSTRACT. The problem of selecting the effective composition and program of trajectory measurements, whose errors are random and independent, is discussed. The concept of the density of measurements of a homogeneous composition (control) is introduced, and the corresponding variational problems are formulated. A proof is presented for the theorem of a non-singular solution in these problems, when there are limitations on the density and the total number of measurements. Singular solutions are studied.

An algorithm is presented for finding the non-singular solution, making it possible to obtain the optimum compositions and measurement programs for specific problems. A solution is presented to the problem of optimal distribution of radial velocity measurements when determining the orbit of an artificial satellite (AS) of Mars.

INTRODUCTION /5

The study [1] presents an algorithm for determining the forecasting accuracy for the case when processing is done by the maximum probability method. It is found that this algorithm may be used in a ballistic flight forecast for selecting the effective composition and program of trajectory measurements. The present article is devoted to this problem. The formulation and solution of the problem are presented below for the optimum selection of the composition and program of measurements, whose errors are assumed to be random and independent.

Elfving [2] made the first study of this problem. In 1962 he investigated this problem for the case of two parameters to be determined. It was assumed that an arbitrarily large number of measurements may be made at a point. The general theorem in a similar formulation of the problem was proven in 1969 by Yermov [3].

In contrast to these studies, this article considers the following features of discrete trajectory measurements: the measurement mass and the limitation on the measurement rate.

Consideration of these characteristics leads to the fact that the extremum solution here essentially differs from the solution which corresponds to theElfving-Yermov theorem (these solutions coincide at the limit).

In addition to the proof for the fundamental theorems, this study presents an algorithm which can be used to obtain the optimum measurement composition and program.

This algorithm is very cumbersome, and it may only be used for calculations on a computer. /6

At the end of the article, we investigate one example connected with determining the optimum measurement program of the radial velocity when determining the orbit of an artificial satellite of Mars.

I. Variational Problems on the Selection of the Optimum Composition and Program of Trajectory Measurements /7

1. Initial Relationships

Let us assume that at the moment $t_1, t_2, \dots, t_N$ measurements are made of a certain function $\Psi(t) = \Psi(t, Q_1, Q_2, \dots Q_m)$ to determine the parameters of motion of a spacecraft. $Q_1, Q_2, \dots Q_m$.

We shall assume that the measurement errors are random, and are subject to a multi-dimensional normal law of distribution, with zero mathematical expectation and known variance. Assuming that the variance of the measurements is constant in time and assuming that it equals unity, we obtain the correlational matrix of the trajectory parameters [1]:

$$K_Q = A^{-1} \tag{1.1}$$

where

$$A = \| A_{ij} \|, \quad A_{ij} = \sum_{K=1}^{N} \frac{\partial \Psi(t_K)}{\partial Q_i} \cdot \frac{\partial \Psi(t_K)}{\partial Q_j}, \quad (i, j = 1, 2, \ldots, m) \tag{1.2}$$

The quantities A and K_Q are symmetrical square matrices.

In this case, the problem may be formulated in the following way. In the time interval $[a, b]$ we may perform N_0 measurements, whose errors are assumed to be random and independent. Let us distribute these measurements for $[a, b]$ in such a way that $[K_Q]_{11}$ (dispersion of the parameter Q_1) is a minimum. /8

Such a measurememt program may be called optimum.

This problem (without limitations on the number of measurements at a point) for $m = 2$ was first investigated in [2]. The general theorem was proven in [3]. The Elfving-Yermov theorem states that the number of optimum measurement subsets (i.e., points with different gradient vectors of the measured function with respect to the parameters being determined) does not exceed the number of parameters.

A solution of this problem is given below for mass measurements when there is a limitation on the number of measurements per unit time.

2. Concepts of Measurement Density

Let us formulate a continuous analog of the matrix elements A from (1.2).

Let us assume $\Psi(t)=\Psi(t, Q_1, Q_2, \dots Q_m)$ $t \in [a,b]$, which is a certain arbitrary function to be measured, which is determined for each selection of the parameters $Q_1, Q_2, \dots Q_m$. In addition, let us assume that for all $t \in [a,b]$ there are isochronous derivatives which are continuous in t:

$$L_i(t)=\frac{\partial \Psi(t)}{\partial Q_i}, \qquad (i=1,2,\dots,m) \tag{1.3}$$

With respect to matrix K_Q from (1.1), it is assumed that it is stable. By stating that K_Q is stable, we mean that small deviations of the matrix elements A lead to small changes of the matrix elements K_Q, which is reciprocal to A [4]. /9

Let us investigate two functions $f_{ij}(t)=L_i(t)\,L_j(t)$ (i,j — fixed) and $N(t)$ for $[a,b]$. According to definition, the function $f_{ij}(t)$ given above is a continuous function $t \in [a,b]$. The function $N(t)$, $t \in [a,b]$ is introduced as a certain continuous analog of the measurement number. Thus, the number of measurements equalling $N(t_k)$, performed in the interval $[a, t_k]$, corresponds to any moment of measurement time $t=t_k$. In the intervals between measurements, the function $N(t)$ is assumed to increase monotonically (in a wide sense). Thus, in contrast to the actual number of measurements, the function $N(t)$ assumes all values in the permissible range of variation in the number of measurements $[0,N_0]$. It is also assumed that N(t) $t \in [a,b]$ has a limited change (the number of measurements is finite).

Let us divide the unknown interval $[a,b]$ by the points $t_0, t_1, \dots, t_n$ as follows: $t_0=a<t_1<t_2 \dots <t_n=b$, and let us examine the function $f_{ij}(t)$ and $N(t)$ corresponding to this division.

Let us assume that $N(t_r)$ is the value of $N(t)$ for the moment of division t_r, and $f_{ij}(\xi_r)$ is the value of $f_{ij}(t)$ at the point $t = \xi_r$, where ξ_r is a certain point in the interval $[t_{r-1}, t_r]$. Let us set $\lambda = \max \Delta t_k$, where $\Delta t_k = t_k - t_{k-1}$, $(k = 1, 2, \dots n)$. Let us calculate the sum

$$S_n^{(ij)} = \sum_{r=1}^{n} f_{ij}(\xi_r)[N(t_r) - N(t_{r-1})] \tag{1.4}$$

and let us investigate its limits, as the number of points of dividing this interval $[a, b]$ approaches infinity. This limit always exists ($f_{ij}(t)$ is continuous in $[a, b]$, and $N(t)$ is a function having a finite change) and is /10
the Stieltjes integral of the function $f_{ij}(t)$ with respect to the function $N(t)$[5]:

$$\lim_{\substack{n \to \infty \\ (\lambda \to 0)}} S_n^{(ij)} = \int_a^b f_{ij}(t)\, dN(t). \tag{1.5}$$

In the further discussions, it will be assumed that almost for all $t \in [a, b]$ there is a finite derivative

$$u(t) = \frac{dN(t)}{dt}, \tag{1.6}$$

and then we arrive at the Riemann integral

$$J_{ij} = \int_a^b f_{ij}(t)\, u(t)\, dt. \tag{1.7}$$

The function $u(t)$, determined by (1.6) characterizes the measurement intensity. In view of this, we shall call $u(t)$, $t \in [a, b]$ the measurement density.

When the indices i, j encompass all values from 1 to m, the integrals (1.7) comprise the matrix

$$J = \| J_{ij} \|, \quad (i, j = 1, 2, \dots m). \tag{1.8}$$

Let us assume that the matrix of small positive elements $\delta = \|\delta_{ij}\|$, $(i,j = 1,2,\ldots,m)$ is given, and that these quantities deviate very little from the elements of the inverse matrix. Then the measurements for which $u(t)$, $t \in [a,b]$ satisfy the inequality

$$\max |A - J| < \delta \tag{1.9}$$

will be called quasi-continuous. /11

Let us note that $u(t)$, $t \in [a,b]$ is a certain smooth function showing the change in the step function $\frac{\Delta N_i}{\Delta t_i}$ $(i=1,2,\ldots,n)$, where ΔN_i is the number of measurements in the interval Δt_i.

It is apparent that this approximation is always valid for mass trajectory measurements. We find from inequality (1.9) and the smallness of the elements of matrix δ that the matrix elements

$$B = J^{-1} \tag{1.10}$$

deviate very little from the elements of the matrix K_Q from (1.1).

The matrices J and B are symmetrical, in view of the symmetry of the matrices A and K_Q.

Below, we shall only study the case of quasi-continuous measurements. For the given function $u(t)$, $t \in [a,b]$ it is assumed that there is a known program of quasi-continuous measurements of the function $\Psi(t, Q_1, Q_2, \ldots, Q_m)$, i.e., the location and the intensity of the measurements are known.

Now let us turn to the problem under consideration. An approximation of the matrix A by means of the matrix J makes it possible to reduce the problem of finding the optimum measurement points to the problem of finding a certain finite function $u(t)$, $t \in [a,b]$, i.e., it is reduced to a non-classical variational problem.

3. Optimum Measurement Program

Let us discuss the problem of finding the optimum program of quasi-continuous measurements of homogeneous composition.

It is clear that any elements of the matrix $\mathcal{J}$ from (1.8) may be regarded as a certain functional of the scalar function $u(t)$, $t \in [\alpha, \beta]$: /12

$$J_{ij}[u(t)] = \int_{\alpha}^{\beta} L_i(t) L_j(t) u(t) dt, \quad (i, j = 1, 2, \ldots, m) \tag{1.11}$$

We shall call the matrix $\mathcal{J}$ the matrix functional of u(t):

$$\mathcal{J} = \mathcal{J}[u(t)] = \| J_{ij}[u(t)] \|, \quad (i, j = 1, 2, \ldots, m). \tag{1.12}$$

Since all the functionals comprising the matrix $\mathcal{J}[u(t)]$ are linear functionals of u(t), we shall call the matrix T[u(t)] a linear square functional of u(t). The inverse matrix $\mathcal{J}[u(t)]$:

$$B[u(t)] = \{\mathcal{J}[u(t)]\}^{-1} \tag{1.13}$$

is also a matrix functional of u(t).

The following theorem holds:

Theorem 1. Let us assume that $L_1(t), \ldots, L_m(t)$ are linearly independent and analytical functions for $[\alpha, \beta]$. In addition, let us assume

$$\det \left\| \int_{\alpha}^{\beta} L_r(t) L_s(t) dt \right\| \neq 0, \quad (r, s = 0, 1, \ldots, m) \tag{1.14}$$

where $L_0(t) \equiv -1$ for $[\alpha, \beta]$.

Let us study the problem of determining the scalar function $u(t)=u^{\circ}(t)$ of the minimizing functional

$$B_{11}[u(t)] = \frac{\det \| J_{kn}[u(t)] \|}{\det \| J_{ij}[u(t)] \|} \ldots, \qquad (1.15)$$ /13

$$(k, n = 2, 3, \ldots m \quad ; \; i,j = 1, 2, \ldots, m)$$

with the limitations

$$0 \leq u(t) \leq c_0 \; , \qquad t \in [\alpha, \beta] \qquad (1.16)$$

$$\int_{\alpha}^{\beta} u(t)dt = c_1 < c_0(\beta - \alpha), \qquad (1.17)$$

where J_{ij}, $(i,j=1,2,\ldots,m)$ are elements of the linear matrix functional (1.12).

Let us define the set $E_+ = E_+[u(t)]$ and $E_- = E_-[u(t)]$ as follows:

$$E_+ = [t : \Gamma(t) + \lambda > 0], \qquad (1.18)$$

$$E_- = [t : \Gamma(t) + \lambda < 0],$$

where

$$\Gamma(t) = [\sum_{i=1}^{m} \frac{B_{1i}}{B_{11}} L_i(t)]^2, \qquad (1.19)$$

The quantities $B_{11}, B_{12}, \ldots B_{1m}$ are elements of the first row of matrix (1.13) for the control $u(t)$, and λ is selected from the condition:

$$c_0 \; mes \; E_+ = c_1 \qquad (1.20)$$

Then the minimizing function $u^{\circ}(t)$ is determined by the following relationships:

$$u^{\circ}(t) = c_0 \quad \text{for} \quad E_+^{\circ} = E_+[u(t) = u^{\circ}(t)],$$

$$u^{\circ}(t) = 0 \quad \text{for} \quad E_-^{\circ} = E_-[u(t) = u^{\circ}(t)]. \qquad (1.21)$$

Let us investigate certain conditions of theorem 1. The limitation (1.16) follows from the definition of the scalar function $u(t), t \in [\alpha, \beta]$, according to which this function can not be negative. In addition, the number of measurements per unit time is always finite (apparatus limitation). In view of (1.6), limitation (1.17) is a limitation on the number of measurements. It is assumed that C_1 is less than $C_0(\beta - \alpha)$ — the maximum number of measurements which can be performed for $[\alpha, \beta]$.

To prove theorem 1, we derive the first variation of the functional (1.15) and study it. We should note that the method of solving the non-classical variational problems, which is based on this principle, is used in studies [6, 7].

Let us first introduce certain definitions. Let $\mathcal{J}[u(t)]$ be the matrix functional of the scalar function $u(t)$. Then the first variation of the matrix function is determined by the following relationships

$$\delta \mathcal{J}[u(t)] = \|\delta \mathcal{J}_{ij}\|, \qquad (i, j = 1, 2, \ldots, m),$$

i.e., the first variations of the corresponding elements of the matrix functional $\mathcal{J}[u(t)]$ are elements of the matrix $\delta \mathcal{J}[u(t)]$.

It is apparent that the rule governing the variation of the product of the matrix functionals coincides with the rule governing the variations of the product of the functionals.

4. Proof of Theorem 1. /15

To find the first variation of the functional under consideration, we shall first calculate the first variation $B[u(t)]$. By definition of the inverse matrix, we have

$$B[u(t)] \cdot \mathcal{J}[u(t)] = E,$$

where E is the unit matrix. The first variation of this relationship in the vicinity of a certain equation $u(t)$, $t \in [a, b]$ may be written in the form $\delta B J + B \delta J = 0$ $(B = B[u(t)],\ J = J[u(t)])$. Multiplying the last expression on the right by the matrix B, we obtain

$$\delta B = -B \delta J B \tag{1.22}$$

Let us introduce the following matrix function for $[a, b]$

$$L(t) = \| L_i(t) \cdot L_j(t) \|, \qquad (i, j = 1, 2, \ldots, m).$$

Then the first variation of the linear matrix functional J may be written as follows

$$\delta J = \int_a^b L(t) \delta u(t) dt,$$

and due to this (1.22) assumes the following form

$$\delta B = -\int_a^b B L(t) B \delta u(t)\, dt. \tag{1.23}$$

Below, for purposes of convenience, we shall investigate a functional which is the inverse of the unknown functional

$$\Phi_1 = \{B_{11}[u(t)]\}^{-1} \tag{1.24}$$

Taking into account (1.28), we find: /16

$$\begin{aligned} \delta \Phi_1 [u(t)] &= \frac{1}{B_{11}^2} \int_a^b [B L(t) B]_{11} \delta u(t) dt = \\ &= \frac{1}{B_{11}^2} \int_a^b \sum_{i=1}^{m} \sum_{j=1}^{m} B_{1i} L_i(t) L_j(t) B_{j1} \delta u(t) dt \end{aligned} \tag{1.25}$$

Let us introduce the notation

$$\nu_j = \frac{B_{1j}}{B_{11}}, \qquad (j = 2, 3, \ldots, m), \quad (\nu_1 = 1) \tag{1.26}$$

and taking into account the symmetry of the matrix B, we finally obtain

$$\delta\Phi_1 = \int_{a}^{b} \Gamma(t)\delta u(t)\,dt, \tag{1.27}$$

where the function $\Gamma(t)$, $t\in[a,b]$ has the form

$$\Gamma(t) = [\sum_{j=1}^{m} \nu_j L_j(t)]^2 . \tag{1.28}$$

Let us investigate the relationship (1.27) in the vicinity of a certain internal control $u(t)$, $t\in[a,b]$, which satisfies conditions (1.17). The following holds:

Lemma 1. Let us assume the conditions of theorem 1 are satisfied. Then any internal control can be "improved" .

We shall prove this statement.

In view of limitation (1.17), variation $\delta u(t)$ in (1.27) is not free, and is subject to the following condition

$$\int_{a}^{b} \delta u(t)\,dt = 0 \tag{1.29}$$

Let us free ourselves from Condition (1.29). To do this, we represent the variation $\delta u(t)$ in the following form /17

$$\delta u(t) = \delta u'(t) + \delta\lambda \tag{1.30}$$

where $\delta u'(t)$ is an arbitrary variation (variable), and $\delta\lambda$ is a certain constant (constant variation).

A specific constant $\delta\lambda$ is put into correspondence with each arbitrary variation $\delta u'(t)$. Thus, with allowance for (1.30) Condition (1.29) yields

$$\delta\lambda = -\frac{1}{b-a}\int_a^b \delta u'(t)\,dt. \tag{1.31}$$

With allowance for the latter relationships, (1.27) may be written in the form

$$\delta\Phi_1 = \int_a^b \varphi(t)\,\delta u'(t)\,dt; \quad \varphi(t) = \Gamma(t) - \frac{1}{b-a}\int_a^b \Gamma(t)\,dt. \tag{1.32}$$

In view of the conditions of theorem 1, the influence function $\varphi(t)$ may equal zero only at a finite number of points. In actuality, the assumption $\varphi(t) \equiv 0$ for $[a, b]$ leads to the following equation for any control $u^*(t)$:

$$L_1(t) + \nu_2^* L_2(t) + \ldots + \nu_m^* L_m(t) = c^*, \tag{1.33}$$

(the quantities $\nu_2^*, \ldots, \nu_m^*$ are values of (1.26) for $u = u^*(t), t \in [a, b]$, which expresses the linear dependence of the function $L_0(t) \equiv -1, L_1(t), L_2(t), \ldots L_m(t)$ for $[a, b]$. Thus, the constant c^* in (1.33) does not equal zero in view of the linear independence of the isochronous derivatives. /18

However, Equation (1.33) contradicts inequality (1.14), which is a necessary and sufficient condition for the linear independence of the function $L_0(t), \ldots, L_m(t)$ for $[a, b]$ [8]. In view of the analytic nature of $L_k(t), \ldots,$ $L_m(t)$ for $[a, b]$ Equation (1.33) (Equation $\varphi(t) = 0$) can hold only at a finite number of points.

Thus, in the first relationship (1.32) $\delta u'(t)$ is an arbitrary variation, and Condition $\varphi(t) = 0$ holds only for a set of measure zero for all equations $u(t)$, $t \in [a, b]$.

Let us assume $\delta u'(t) = \mu\,\varphi(t)$, $\mu > 0$, $\mu - const$. Then $\delta\Phi_1 > 0$. This means that for the control $u(t) + \delta u(t)$, the functional will be greater than for the control $u(t)$.

Since $u(t)$ is an arbitrary internal control, any internal control may be "improved" (in the sense of satisfying Condition $\delta\Phi_1 > 0$).

Thus, Lemma 1 is proven.

It follows from Lemma 1 that the extremum is achieved at the boundary of the permissible region of control $u(t)$, since any internal control can be "improved" (it is assumed that the extremum of the problem exists).

To construct the extremum control, let us investigate the first variation of the functional of the problem. In view of (1.17), the functional of the problem has the form

$$G[u(t)] = \Phi_1[u(t)] + \lambda \int_a^b u(t)\,dt,$$

where λ is a Lagrangian multiplier. The first variation of this relationship /19
in the vicinity of a certain boundary control has the form

$$\delta G = \int_a^b (\Gamma(t) + \lambda)\,\delta u(t)\,dt \tag{1.34}$$

The Rayleigh control $u^{\circ}(t)$ from (1.21) is an extremum, i.e., for ε in the vicinity of $u^{\circ}(t)$, according to (1.34) we have $\delta G \leq 0$. In actuality for E_+° we have $\delta u^{\circ}(t) \leq 0, \Gamma(t) + \lambda > 0$, and for E_-° - $\delta u^{\circ}(t) \geq 0$, $\Gamma(t) + \lambda < 0$. It is thus assumed that for the variation $\delta u^{\circ}(t)$ the Condition (1.29) holds.

Any other boundary control is not an extremum. In actuality, if we have $u^{\circ}(t) = 0$ at any section E_+°, then $\delta G > 0$, that is, $\Gamma(t) > 0$ and $\delta u(t) > 0$.

Theorem 1 is proven.

Thus, the extremum solution of the problem under Condition (1.14) is reached at boundary of the permissible control region u(t). The total duration of the optimum measurement periods is determined by the constant

$\frac{C_1}{C_0}$ [see (1.20)]. The quantity C_0 influences the variance of the parameters being determined, which decreases with an increase in C_0.

5. Case when Condition (1.14) is Not Satisfied

If a control exists which satisfies the limitations (1.16) and (1.17), for which Equation (1.33) holds, then this control will be called singular.

A control may be singular, if inequality (1.14) is not satisfied. However, this condition is not sufficient, in view of the specific nature of the coefficients having linear form (1.33). /20

Let us first obtain certain relationships following from Equation (1.33).

In this equation, the quantities $\nu_2^*, \nu_3^*, \ldots \nu_m^*$ may be represented as follows

$$\nu_j^* = \frac{B_{1j}^*}{B_{11}^*} = \frac{D_{1j}^*}{D_{11}^*}, \qquad (j=2,3,\ldots,m) \tag{1.35}$$

where $D_{11}^*, D_{12}^*, \ldots D_{1m}^*$ are the cofactors which correspond to the first row of the matrix $\mathcal{J}^*$ with the elements

$$\mathcal{J}_{ij}^* = \int_a^b L_i(t) L_j(t) u^*(t)\,dt, \qquad (i,j=1,2,\ldots,m) \tag{1.36}$$

Then (1.33) may be written in the form

$$D_{11}^* L_1(t) + D_{12}^* L_2(t) + \ldots + D_{1m}^* L_m(t) = D_{11}^* c^* \tag{1.37}$$

Let us perform integration m times in the limits a, b of relationship (1.37), multiplied respectively by the quantities $L_2(t)u^*(t), L_3(t)u^*(t), \ldots L_m(t)u^*(t), u^*(t)$. With allowance for the notation in (1.36) we obtain the following relationships

$$
\begin{aligned}
&\sum_{j=1}^{m} D_{1j}^{*} J_{2j}^{*} = D_{11}^{*} c^{*} \int_{a}^{b} L_2(t)\, u^{*}(t)\, dt, \\
&\sum_{j=1}^{m} D_{1j}^{*} J_{3j}^{*} = D_{1}^{*} c^{*} \int_{a}^{b} L_3(t)\, u^{*}(t)\, dt, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\sum_{j=1}^{m} D_{1j}^{*} J_{mj}^{*} = D_{11}^{*} c^{*} \int_{a}^{b} L_m(t)\, u^{*}(t)\, dt, \\
&\sum_{j=1}^{m} D_{1j}^{*} \int_{a}^{b} L_j(t)\, u^{*}(t)\, dt = c^{*} D_{11}^{*} c_1 .
\end{aligned}
\tag{1.38}
$$

The left sides of the first $(m-1)$ relationships (1.38) contain the /21
product of the cofactors of elements in the first row of the matrix J^{*} by elements of the second, third, etc. m^{th} row of the same matrix. All these products equal zero according to the specific property of the determinant. Finally, taking into account condition $c^{*} \neq 0$, we obtain

$$\int_{a}^{b} L_j(t)\, u^{*}(t)\, dt = 0, \qquad (j = 2, 3, \dots m). \tag{1.39}$$

We should note that condition $D_{1}^{*} \neq 0$ is also assumed, which is always valid for a finite number of measurements and linearly independent isochronous derivatives.

The last relationship in (1.38), with allowance for (1.39), gives

$$\int_{a}^{b} L_1(t)\, u^{*}(t)\, dt = c^{*} c_1 . \tag{1.40}$$

For the case when condition (1.14) is not satisfied, the following statement is valid:

<u>Theorem 2</u>. Let us assume $L_1(t), L_2(t), \dots, L_m(t)$ - are linearly independent functions on $[a, b]$ which satisfy the equation

$$\det \left\| \int_{a}^{b} L_k(t)\, L_s(t)\, dt \right\| = 0, \qquad (k, s = 0, 1, 2, \dots m) \tag{1.41}$$

where $L_0(t) \equiv -1$ на $[a, b]$. /22

Let us consider the problem of determining the scalar function $u(t)$, $t \in [a, b]$ which minimizes the functional (1.15) for the limitations (1.16) and (1.17).

Let us assume one control $u^*(t)$, $t \in [a, b]$ exists which satisfies the conditions (1.16), (1.17) and Equations (1.39).

Then this control provides the extremum of the problem being considered, and will belong to a class of singular controls. The set of singular controls is infinite. The functional (1.15) of this set is constant.

In the opposite case, when only Condition (1.41) holds, the extremum control for the problem being considered will be non-singular and will be determined according to (1.21).

Proof

It follows from Condition (1.41) that for $[a, b]$ the following equation holds

$$\alpha_1 L_1(t) + \alpha_2 L_2(t) + \ldots + \alpha_m L_m(t) = \alpha_0 , \tag{1.42}$$

where α_j $(j = 0, 1, 2, \ldots m)$ do not equal zero simultaneously [8]. Thus, $\alpha_0 \neq 0$ in view of the linear independence of $L_1(t), \ldots, L_m(t)$. We shall show that $\alpha_1 \neq 0$. To do this, we shall integrate Relationship (1.42) within the limits $[a, b]$. This relationship multiplied by $u^*(t)$ is the control satisfying the Conditions (1.39):

$$\alpha_1 \int_a^b L_1(t)\, u^*(t)\, dt = \alpha_0 c_1 , \tag{1.43}$$

where c_1 is the constant sum Condition (1.17). Thus $\alpha_1 \neq 0$. /23

Let us divide Relationship (1.42) by $\alpha_1 \neq 0$. We obtain

$$L_1(t) + \beta_2 L_2(t) + \ldots + \beta_m L_m(t) = \beta_0,$$

where $\beta_i = \frac{\alpha_i}{\alpha_1}$, $\quad (i = 0, 2, 3, \ldots, m)$.

Let us show that $\beta_j = \nu_j^*$, $(j = 2, 3, \ldots m)$. To do this, we shall investigate the cofactor D_{12}^*. The first column of the determinant D_{12}^* is transformed, with allowance for the equation $L_1(t) = \beta_0 - \beta_2 L_2(t) - \ldots - \beta_m L_m(t)$ following from (1.44).

We have

$$D_{12}^* = (-1)^3 \begin{vmatrix} \beta_0 \int_a^b L_2 u^* dt - \beta_2 \int_a^b L_2^2 u^* dt - \ldots - \beta_m \int_a^b L_2 L_m u^* dt & J_{23}^* & \ldots & J_{2m}^* \\ \beta_0 \int_a^b L_3 u^* dt - \beta_2 \int_a^b L_3 L_2 u^* dt - \ldots - \beta_m \int_a^b L_3 L_m u^* dt & J_{33}^* & \ldots & J_{3m}^* \\ \ldots & \ldots & \ldots & \ldots \\ \beta_0 \int_a^b L_m u^* dt - \beta_2 \int_a^b L_m L_2 u^* dt - \ldots - \beta_m \int_a^b L_m^2 u^* dt & J_{m3}^* & \ldots & J_{mm}^* \end{vmatrix}$$

With allowance for relationships (1.39), the first terms in the first column of D_{12}^* disappear. Finally, changing to the notation (1.36), we find that the remaining terms in the first column of D_{12}^* contain a linear combination of the following columns of the determinant D_{12}^*. Taking this fact into account, we obtain $D_{12}^* = \beta_2 D_{11}^*$, i.e., $\beta_2 = \nu_2^*$. Similarly, we may show that $\beta_3 = \nu_3^*, \ldots, \beta_m = \nu_m^*$. (To prove these equations, it is necessary to interchange the columns in the corresponding determinants). /24

Thus, Relationships (1.33) and (1.44) have the same left sides and, consequently, the same right sides, i.e., $u^*(t)$ — is a singular control. However, since Condition $\Gamma(t) + \lambda = 0$ for $[a, b]$ corresponds to (1.33) in (1.32) we have $\delta\Phi_1 = 0$, i.e., $u^*(t)$ satisfies the necessary condition of the extremum.

Let certain values of $\nu_2^*, \ldots, \nu_m^*$ $(c_1 = \int_a^b u^* dt)$ correspond to the singular control $u^*(t)$. Any control which provides the same values to the quantities in (1.35) will also be singular.

Let us formulate the problem of determining the entire set of controls from the Conditions $\nu_2 = \nu_2^*, \ldots, \nu_m = \nu_m^*$ and the limitations (1.16), (1.17), where $\nu_2^*, \ldots \nu_m^*$ are given constants, and $\nu_2, \ldots, \nu_m$ are determined by (1.26). Under the given conditions and the limitation (1.17), the control is only included under the indices of specific integrals. The integrands are functions of one variable (the integration variable). Thus, the number of singular controls satisfying the limitations (1.16), (1.17) will be infinite.

The functional is constant on an infinite set of singular controls. In actuality, we find the following from Conditions (1.24), (1.15), (1.33), and (1.40):

$$\Phi_1^* = \frac{|J^*|}{D_{11}^*} = \int_a^b L_1(t)\Big(\sum_{j=1}^m L_j^* L_j(t)\Big) u^*(t) dt = c^* \int_a^b L_1 u^* dt = (c^*)^2 c_1.$$ /25

It is apparent that in the remaining cases, if only Condition (1.41) holds, and Condition (1.39) is not satisfied for any control $u(t)$, $t \in [a, b]$ satisfying the limitations (1.16), (1.17), then we arrive at the linear form (1.42). This form does not coincide with the form of (1.33), i.e., the control will not be singular and is determined according to (1.21).

The theorem 2 is proven.

Corollary. If $u^*(t)$, $t \in [a, b]$ is singular, then with $L_1(t) = const$ for $[a, b]$ $(m > 1)$ we have $\nu_2^* = \nu_3^* = \ldots = \nu_m^* = 0$.

In actuality, all the cofactors $D^*_{12}, D^*_{13}, \ldots D^*_{1m}$ have one and the same first column with the elements $J^*_{j1}\ (j=2,3,\ldots m)$. This column will be zero with $L_1(t)=const$, in view of the fact that $J^*_{j1}=L_1\int_a^b L_j u^* dt=0, \quad (j=2,3,\ldots,m)$, as follows from Raltionship (1.39). Finally, taking into account (1.35), we obtained the desired corroboration.

The simplest example of a singular control pertains to the case: $m=1$, $\Psi(t)=Q_1$, $t\in[a,b]$. Since $L_1(t)\equiv 1$ for $[a,b]$, any control $u^*(t), t\in[a,b]$ satisfying the Conditions (1.16), (1.17) provides one and the same value $\Phi_1=C_1$ to the functional Φ_1. This conclusion is obvious, since when there is one parameter and its indirect measurement is performed, it does not matter when an infinite number of measurements is carried out. /26

We could give many examples of singular controls for m > 1. However, as a rule, they are only of a mathematical character.

However, a singular control may occur in practical problems. For example, let us investigate a navigational problem connected with determining two parameters characterizing the position of the plane of an elliptic orbit. A detailed description of this problem is given in [9]. In [9] a special case is studied in a discrete formulation. For brevity, we shall give this problem in a continuous formulation.

We have: $\Psi(t)=Q_1\cos\theta(t)+Q_2\sin\theta(t), \ t\in[a,b]$, , ($\theta(t)=\theta$ is the true anomaly). The sum of the variances is minimized: $\Phi = B_{11} + B_{22}$. The absolute minimum Φ is achieved under the condition that relationships $\int_a^b \cos 2\theta\, u^*(t)dt=\int_a^b \sin 2\theta\, u^*(t)dt=0$ are satisfied if $u^*(t), t\in[a,b]$.

For example, for $\theta\in[0,\pi]$ we can readily select an infinite set of controls $u^*(t)$ satisfying Conditions (1.16) and (1.17).

An equation similar to (1.33) is satisfied in this case, since the coefficients will equal zero for $L_1(t)$ and $L_2(t)$ when the relationships given above are satisfied. /27

6. Variational Problem for Some Measurement Compositions

On the interval $[a, b]$ let us give $M>1$ functions to be measured $\psi^{(1)}(t), \dots \psi^{(M)}(t)$, each of which depends on m parameters of the orbit $Q_1, Q_2, \dots, Q_m$. In addition, let us give the variances $\sigma_1^2, \dots, \sigma_m^2$ for each measured function as certain constants. For measurements whose errors are random and independent, the correlational matrix of the parameters to be determined is as follows

$$K_Q = \left\{ \sum_{z=1}^{M} A^{(z)} \right\}^{-1}, \tag{1.45}$$

where $A^{(z)}$ is the weight matrix for the z^{th} composition of the measurements, which has the following form by analogy with (1.2)

$$A^{(z)} = \| A_{ij}^{(z)} \|, \quad A_{ij}^{(z)} = \sum_{k=1}^{N_z} \frac{\partial \psi^{(z)}(t_k)}{\partial Q_i} \cdot \frac{\partial \psi^{(z)}(t_k)}{\partial Q_j} \cdot \frac{1}{\sigma_z^2}, \quad (i,j=1,2,\dots m) \tag{1.46}$$

where N_z is the number of measurements of the z^{th} composition.

Let us introduce the equation

$$L_i^{(z)}(t) = \frac{\partial \psi^{(z)}(t)}{\partial Q_i}, \quad (i=1,2,\dots m;\ z=1,\dots M) \tag{1.47}$$

The distribution of the quasi-continuous measurements in this case is determined by the vector function

$$U(t) = \{ u_1(t), \dots u_M(t) \}, \tag{1.48}$$

each component of which is the density of the measurements of the corresponding composition.

For the quasi-continuous measurements, we obtain

$$\mathcal{J}=\sum_{s=1}^{M}\mathcal{J}^{(s)},\quad \mathcal{J}^{(s)}=\left\|\int_{a}^{b}L_i^{(s)}(t)L_j^{(s)}(t)\,u_s(t)\,dt\right\|,\ (i,j=1,2,..m;\ s=1,...M) \tag{1.49}$$

We should note that the matrix $\mathcal{J}$ from (1.49) may be regarded as the matrix functional of the control vector $V(t),\ t\in[a,b]$: $\mathcal{J}=\mathcal{J}[V(t)]$.

For $M>1$ measurement compositions, we may also formulate the problem of optimizing the observation process.

Separate determination at the parameters will not be considered, i.e., when the compositions are not given by means of the parameters being determined. If this case occurs, the problem of simultaneously determining the optimum composition and measurement program is devoid of any meaning.

For the problem of determining the optimum composition and measurement program, the following theorem may be proven:

<u>Theorem 3</u>. For $[a,b]$, let us give $M>1$ systems of analytical functions: $L_1^{(1)}(t),...L_m^{(1)}(t);\ ...\ L_1^{(M)}(t),...L_m^{(M)}(t)$. Let the elements of each system be linearly independent for $[a,b]$, and satisfy the conditions

$$\det\left\|\int_{a}^{b}L_k^{(s)}L_z^{(s)}\,dt\right\|\neq 0,\qquad L_0^{(s)}(t)\equiv -1,\ t\in[a,b], \tag{1.50}$$

$$(s=1,\ ...\ M;\quad k,z=0,1,...m)$$

Let us investigate the problem of determining the control vector /29
$V(t)=V^0(t),\ t\in[a,b]$ which minimizes the functional

$$B_{11}\,[V(t)]=(\{\mathcal{J}[V(t)]\}^{-1})_{11} \tag{1.51}$$

under the limitations

$$0\leq u_s(t)\leq c_0^{(s)},\qquad (s=1,2,...M) \tag{1.52}$$

$$\int_{a}^{b}\Big[\sum_{s=1}^{M} u_s(t)\Big]dt = C_1. \tag{1.53}$$

Let us determine the sets $E_+^{(s)} = E_+^{(s)}[U(t)]$ and $E_-^{(s)} = E_-^{(s)}[U(t)]$, $(s = 1,2,\dots M)$ as follows:

$$\begin{aligned} E_+^{(s)} &= [t : \Gamma^{(s)}(t) + \lambda > 0], \\ E_-^{(s)} &= [t : \Gamma^{(s)}(t) + \lambda < 0], \\ &\qquad (s = 1,2,\dots M) \end{aligned} \tag{1.54}$$

where

$$\Gamma^{(s)}(t) = \Big[\sum_{i=1}^{m} \frac{B_{1i}}{B_{11}} L_i^{(s)}(t)\Big]\frac{1}{\sigma_s^2}, \quad (s = 1,2,\dots,M),$$

$B_{11}, B_{12}, \dots B_{1m}$ are elements of the first row of matrix $B[U(t)] = \{J[U(t)]\}^{-1}$, and the value λ is selected from the condition

$$C_1 = C_0^{(1)} \,\mathrm{mes}\, E_+^{(1)} + \dots + C_0^{(M)} \,\mathrm{mes}\, E_+^{(M)}. \tag{1.55}$$

Then the components of the minimizing vector function $U^0(t)$ are determined by the relationships /30

$$\begin{aligned} u_s^0 &= c_0^{(s)} \quad \text{на} \quad (E_+^{(s)})^0 = E_+^{(s)}[U(t) = U^0(t)], \\ u_s^0 &= 0 \quad \text{на} \quad (E_-^{(s)})^0 = E_-^{(s)}[U(t) = U^0(t)]. \\ &\qquad (s = 1,2,\dots M) \end{aligned} \tag{1.56}$$

Let us consider certain conditions of theorem 3.

Here Condition (1.53) is a limitation on the total number of measurements. We may formulate the problem of when, instead of limitation (1.53), $M > 1$ limitations exist of the form (1.17). This problem will be similar to the preceding problem. In this case, limitation (1.53) makes it possible to "transfer" the measurements from one composition to another, and to select the optimum measurement composition, in addition to the optimum program.

The theorem 3 may be proved by the method presented above. However, there are certain features which we must discuss.

Instead of B_{11}, the inverse functional $\Phi_M = \{B_{11}[U(t)]\}^{-1}$ from (1.51) is examined. The first variation of this functional in the vicinity of the values of a certain control vector $U(t)$ has the form

$$\delta\Phi_M = \int_a^\ell \Big[\sum_{s=1}^{M} \Gamma^{(s)}(t)\,\delta u_s(t)\Big]dt, \tag{1.57}$$

where: /31

$$\Gamma^{(s)}(t) = \Big[\sum_{j=1}^{m} \nu_j L_j^{(s)}(t)\Big]^2 \frac{1}{\sigma_s^2}, \quad \nu_j = \frac{B_{1j}[U(t)]}{B_{11}[U(t)]} \tag{1.58}$$

The following holds:

<u>Lemma 2</u>. Let the conditions of theorem 3 be satisfied. Then any internal control $U(t)$ may be "improved".

To prove this, let us proceed as we did in section 4. In view of the integral limitation of the problem, variations $\delta u_1(t), \ldots \delta u_M(t)$ are related by the condition:

$$\int_a^\ell \Big[\sum_{s=1}^{M} \delta u_s(t)\Big]dt = 0. \tag{1.59}$$

Let us make the substitution

$$\begin{aligned} \delta u_1(t) &= \delta v_1(t) + \delta\lambda, \\ &\cdots\cdots\cdots\cdots \\ \delta u_M(t) &= \delta v_M(t) + \delta\lambda, \end{aligned} \tag{1.60}$$

where $\delta V_1(t), \ldots \delta V_M(t)$- are arbitrary variations, and $\delta\lambda$ is a certain constant. We find the following from the latter relationships

$$\delta\lambda = -\frac{1}{M(b-a)} \int_a^b \Big[\sum_{s=1}^{M} \delta V_s(t)\Big] dt, \tag{1.61}$$

due to which the first variation of the desired functional may be finally written in the form /32

$$\delta\Phi_M = \int_a^b \Big\{\sum_{s=1}^{M} [\Gamma^{(s)}(t) + R]\delta V_s(t)\Big\} dt; \quad R = -\frac{1}{M(b-a)} \int_a^b \Big[\sum_{s=1}^{M} \Gamma^{(s)}(t)\Big] dt. \tag{1.62}$$

In view of the fact that $\delta V_1(t), \ldots, \delta V_M(t)$ are arbitrary variations, we may show (just as in section 4) that any internal control $V(t)$ may be "improved".

Thus, for all permissible controls, the functions $\Gamma^{(s)}(t) + R$, $(s = 1, 2, \ldots, M)$ are zero for $[a, b]$ at a finite number of points, in view of the analytic nature of $L_k^{(s)}$, $(s = 1, 2, \ldots M;\ k = 1, 2, \ldots m)$ and Condition (1.50).

Thus, the extremum control is achieved as a boundary of the permissible region.

The fact that the boundary control (1.56) is an extremum may be proven just as in section 4 [control (1.56) cannot be "improved"].

It follows from (1.56) that the control vector $V^{\circ}(t)$ depends on one constant λ. This may lead to the situation when one or several components of the control vector $V^{\circ}(t)$ are identically zero at the interval $[a, b]$. This will mean that the measurements of these compositions are not effective in the sense of the minimum of the functional (1.51). Therefore, in this problem the optimum measurement compositions are selected simultaneously with the optimum measurement programs.

The special cases in this problem may hold only when conditions (1.50) are violated. However, satisfying the equations in (1.50) does not mean that the control is singular: /33

The control satisfying the following conditions will be singular

$$\tilde{\Gamma}^{(s)}(t) + \tilde{R} \equiv 0, \quad t \in [\alpha, \beta], \quad (s = 1, 2, \ldots m) \tag{1.63}$$

or

$$\frac{1}{|\sigma_s|} \sum_{j=1}^{m} \tilde{\nu}_j \tilde{L}_j^{(s)}(t) = \tilde{c}, \quad t \in [\alpha, \beta], \quad (s = 1, 2, \ldots, m) \tag{1.64}$$

where the sign $(\sim)$ shows that these conditions hold for $V(t) = \tilde{V}(t)$, $t \in [\alpha, \beta]$.

Since one and the same constant is on the right hand side of (1.64), it thus follows that the isochronous derivatives for different measurement compositions are linearly dependent.

It is usually assumed that this does not occur. In view of this assumption, an equation like (1.64) can hold only for any one function to be measured.

For this case, the number of singular controls $\tilde{V}(t)$ will be infinite.

In conclusion, let us note that, in the analysis of the solutions to the variational problems, conditions for the analytic nature of the isochronous derivatives were assumed. These conditions must be valid for the solution of each specific problem. This validity may be readily achieved, in particular when studying unperturbed Kepler motion, since in this case there are known expansions of the coordinates and velocities in series, and as a rule they are used to calculate the desired derivatives [10]. /34

II. Algorithm for Determining a Non-singular Solution

1. System of Transcendental Equations

Let us study a known singular solution of the variational problem [see (1.21)] for the case of a homogeneous measurement composition.

In view of the Rayleigh extremum control in specific problems, it is necessary to find the measurement periods when $u(t)=C_0$. In the remaining sections of the interval $[a, b]$, measurements are not performed. When the number of points for shifting the control is known (the boundaries of $[a, b]$, if they are a solution, are also assumed to be shifting points) the solution to the problem does not entail great difficulties. However, as a rule, this number is unknown, and must automatically follow from the solution of the specific extremum problem. Below, we shall study this case.

Let us show that the solution of the given extremum problem may be reduced to finding a finite number of independent variables in a special system of transcendental equations.

As the independent variables, let us use the quantities $\nu_2, \nu_3, \dots \nu_m$ $(\nu_1 \equiv 1)$, which determine the function $\Gamma(t)$. To compile this system, let us first consider the following sequence of operations $(C_0=1)$:

(a) Formulation of the function $\Gamma(t)=[\sum_{j=1}^{m} \nu_j \cdot L_j(t)]^2$, $t \in [a, b]$, $\nu_1 \equiv 1$.

(b) Determination of the points $t_{2\ell-1}$, $t_{2\ell}$ $(\ell=1,2,\dots k)$ as boundaries $k \geq 1$ of the intervals of time satisfying the conditions

$$\begin{gathered} \Gamma(t)+\lambda \geq 0, \qquad t \in [a, b] \\ \sum_{\ell=1}^{k} (t_{2\ell}-t_{2\ell-1})=C_1 . \end{gathered} \tag{2.1}$$

(c) Calculating the elements of the matrix $\mathcal{J}$:

$$\mathcal{J}=\left\|\sum_{\ell=1}^{k}\int_{t_{2\ell-1}}^{t_{2\ell}}L_i(t)L_j(t)dt\right\|,\qquad (i,j=1,2,\dots m) \tag{2.2}$$

(d) Obtaining the following quantities:

$$\tilde{\nu}_j=\frac{B_{1j}}{B_{11}},\qquad (j=2,3,\dots m), \tag{2.3}$$

where $B_{11}, \dots B_{1m}$ are elements of the matrix $B=\mathcal{J}^{-1}$.

It follows from the necessary condition of a strong, relative extremum, that if $\nu_j=\tilde{\nu}_j$ ($j=2,3,\dots m$, then the points obtained from (2.1) determine the Rayleigh control corresponding to the solution of the variational problem.

Thus, to find the extremum solution of the variational problem, it is necessary to determine the independent variables $\nu_2, \dots \nu_m$ from the solution of the following special operator system of equations:

$$\begin{cases} \nu_2=\tilde{\nu}_2(\nu_2,\nu_3,\dots \nu_m),\\ \nu_3=\tilde{\nu}_3(\nu_2,\nu_3,\dots \nu_m),\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \nu_m=\tilde{\nu}_m(\nu_2,\nu_3,\dots \nu_m); \end{cases} \tag{2.4}$$ /37

where the right-hand sides are obtained from the sequence of procedures (a) - (d) given above.

Let us dwell in detail on Conditions (2.1). The points $t_1, t_2, \dots t_{2k}$ which satisfy these conditions will be called the points of switching the control. They are uniquely determined from Conditions (2.1) in view of the analytic nature of the isochronous derivatives and the condition $\Gamma(t)+\lambda\neq 0$ on $[\alpha,\beta]$.

This determination may be made in the general case on a computer. Thus, for the assumed values of $\nu_2, \nu_3, \ldots \nu_m$ $(\nu_1 \equiv 1)$ the functions $\Gamma(t)$, $t \in [a, b]$ are constructed. Then we select $|_{(0)}\nu|$ — an arbitrary initial approximation for $|\lambda|$, contained between zero and $\Gamma^* = \max_{a \le t \le b} \Gamma(t)$. (For $|\lambda| = 0$ we have $mes\, E_+ = c_0 (b-a)$; for $|\lambda| = \Gamma^* - mes\, E_+ = 0$).

The points $\kappa^{(0)} \geq 1$ which are boundaries of the $|\lambda^{(0)}|$ time intervals, for which the conditions $\Gamma(t) \geq |\lambda^{(0)}|$, $t \in [a, b]$ are satisfied, correspond to the points $t_1^{(0)}, t_2^{(0)}, \ldots t_{2\kappa^{(0)}}^{(0)}$. In the general case, these points will be zero functions $\Gamma(t) - |\lambda^{(0)}|$ for $[a, b]$. In addition, the set $2\kappa^{(0)}$ of points may include the boundaries $[a, b]$, if $\Gamma(a) > |\lambda^{(0)}|$, and $\Gamma(b) > |\lambda^{(0)}|$ (or one of the boundaries when only one inequality is satisfied). /38

We should note that when the roots of the function $\Gamma(t) - |\lambda^{(0)}|$ are found for $[a, b]$, it is assumed that their number is unknown. The roots of this function may be determined on a computer. For this purpose, a table of values for the function $\Gamma(t)$, $t \in [a, b]$ is compiled with a certain step h (which is selected empirically). In addition, points are found where $\Gamma(t) - |\lambda^{(0)}|$ changes sign. These points are assumed as zero approximations of the roots. The roots of equation $\Gamma(t) - |\lambda^{(0)}| = 0$ are found with the given accuracy by means of the analytic expression for $\Gamma(t)$. Then the same procedure is repeated for $\frac{h}{2}$. If the number of corresponding roots for $\frac{h}{2}$ does not change, the roots obtained are assumed to be the desired ones.

A change in the number of roots for $\frac{h}{2}$ means that this procedure must be repeated. This is continued until the required condition is satisfied.

Let us set $c_1^{(0)} = \sum_{l=1}^{\kappa^{(0)}} [t_{2l}^{(0)} - t_{2l-1}^{(0)}]$. If $c_1^{(0)} < c_1$, then, to satisfy the integral limitation of the problem, we select a new value of λ from the following considerations: $0 < |\lambda^{(1)}| < |\lambda^{(0)}|$, however, if $c_1^{(0)} > c_1$, then $|\lambda^{(0)}| < |\lambda^{(1)}| < \Gamma^*$. In additi n, for this value $|\lambda| = |\lambda^{(1)}|$, $\kappa^{(1)}$ time intervals are found wiich satisfy the condition $\Gamma(t) \geq |\lambda^{(1)}|$, $t \in [a, b]$. In the general case, $\kappa^{(1)}$ does not

coincide with $K^{(s)}$. Then the value of $c_1^{(1)}$ is calculated which is compared with c_1. This process is terminated on the s^{th} step, when we arrive at the inequality /39
$|c_1^{(s)}-c_1|<\varepsilon_{c_1}$, where ε_{c_1} is the assumed accuracy. The values of $t_1^{(s)}, \dots,$ $t_{2K^{(s)}}^{(s)}$ are used as the points of switching for calculating the matrix (2.2).

Now let us investigate the extremum problem for $M>1$ measurement compositions. It may be readily seen that the solution of the problem is also reduced to the system (2.4). However, the right-hand sides of this system will be formed from the following sequence of procedures (see theorem 3):

(a) Compiling M functions which determine the points of switching the control in the interval $[\alpha, \beta]$:

$$\Gamma^{(s)}(t)=\frac{1}{\sigma_s^2}\left[\sum_{j=1}^{m}\nu_j L_j^{(s)}(t)\right]^2, \quad (s=1,2,\dots M), \quad \nu_1=1.$$

(b) Finding M sets of points for switching the control, as boundaries of the time intervals satisfying the conditions

$$\Gamma^{(1)}(t)+\lambda\geq 0, \qquad \Gamma^{(2)}(t)+\lambda\geq 0, \quad \dots \Gamma^{(M)}(t)+\lambda\geq 0, \; t\in[\alpha,\beta]$$
$$\sum_{s=1}^{M}\sum_{\ell=1}^{K_s}\left(t_{2\ell}^{(s)}-t_{2\ell-1}^{(s)}\right)c_0^{(s)}=c_1, \tag{2.1'}$$

where K_s is the number of time intervals corresponding to $E_+^{(s)}$, $(s=1,2,\dots M)$.

(c) Formulating the matrix

$$J[U(t)]=\sum_{s=1}^{M}\frac{1}{\sigma_s^2}J^{(s)}, \tag{2.2}$$

where /40

$$\mathcal{J}^{(s)} = \left\| \sum_{\ell=1}^{K_s} \int_{t_{2\ell-1}^{(s)}}^{t_{2\ell}^{(s)}} L_i^{(s)}(t) L_j^{(s)}(t)\,dt \right\|, \quad (s=1,2,\dots M;\ i,j=1,2,\dots m).$$

(d) Calculating the right-hand sides of system (2.4)

$$\tilde{\nu}_j = \frac{\{B[U(t)]\}_{1j}}{\{B[U(t)]\}_{11}}, \qquad B[U(t)] = \{J[U(t)]\}^{-1} \tag{2.3'}$$

It follows from the above formulas that the order of the system (2.4) equals ($m-1$), i.e., it is only determined by the number of parameters to be established, and does not depend on the amount of points for switching the control or the selection of the functions to be measured.

2. A Modified Method of Steepest Descent

To solve system (2.4), we shall use the modified method of steepest descent which was proposed by T. M. Eneyev in 1957. This method is described in the article [11] and [1].

The solution of system (2.4) is equivalent to the problem of finding the minimum of the functional

$$\Phi = \xi_2^2 + \xi_3^2 + \dots + \xi_m^2, \tag{2.5}$$

where

$$\xi_j = \nu_j - \tilde{\nu}_j, \qquad (j=2,3,\dots m). \tag{2.6}$$

This minimum is achieved for $\xi_j^* = 0$, $(j=2,3,\dots m)$, which corresponds to the solution of system (2.4). /41

Let us consider the case when the partial derivatives of Φ with respect to the independent variables $\nu_2, \nu_3, \ldots \nu_m$ are continuous functions of the latter (the analytical expressions for these derivatives are given below). The steepest descent thus corresponds to the direction of the anti-gradient [12].

The surface Φ in the space of the variables $\xi_2, \xi_3, \ldots \xi_m$ has a very simple form [1]. This is none other than a "paraboloid of revolution" in m-dimensional space. Correspondingly, steepest descent over the surface of this paraboloid represents the motion along its generatrix.

The gradient descent in this space leads to the system of equations [1]

$$\frac{d\xi_i}{ds} = -\frac{\xi_i}{\sqrt{\Phi}}, \qquad (i = 2, 3, \ldots m) \tag{2.7}$$

or, using the rule of differentiation of complex functions, we obtain

$$a_{i2}\frac{d\nu_2}{ds} + a_{i3}\frac{d\nu_3}{ds} + \ldots + a_{im}\frac{d\nu_m}{ds} = -\frac{\xi_i}{\sqrt{\Phi}}, \quad (i = 2, 3, \ldots m) \tag{2.8}$$

where $a_{ij} = \frac{\partial \xi_i}{\partial \nu_j}$ and the matrix $\bar{a} = \| a_{ij} \|$ must be non-singular.

The numerical integration of the system of Equations (2.8) is usually replaced by an equivalent procedure — the Newton iteration cycle. Thus, at each step, the following system of linear equations is solved, which is generated by the system (2.8) /42

$$a_{i2}\Delta\nu_2 + a_{i3}\Delta\nu_3 + \ldots + a_{im}\Delta\nu_m = -\xi_i, \quad (i = 2, 3, \ldots m) \tag{2.9}$$

To go on to the following step, we must satisfy the conditions

$$\Phi\left(\nu + \frac{1}{2^{\kappa_0}}\Delta\nu\right) < \Phi(\nu), \tag{2.10}$$

where $k_0 = \min\{0, 1, 2, \dots\}$.

3. Derivatives of the Right-hand Sides of the System of Transcendental Equations

Coefficients of the system (2.9) comprise the matrix $\bar{a} = \| a_{ij} \|$, $(i, j = 2, 3, \dots m)$. Let us present a brief derivation for the elements of this matrix for the system (2.4). We have

$$\bar{a} = E - N, \tag{2.11}$$

where E is the unit matrix of order $(m-1)$,

$$N = \left\| \frac{\partial \tilde{\nu}_i}{\partial \nu_j} \right\|, \qquad (i, j = 2, 3, \dots m) \tag{2.12}$$

Let us consider the case of the homogeneous composition of measurements. /43
We should note that for each given set $\nu_2, \nu_3, \dots \nu_m$ the points for switching the control $t_1, t_2, \dots t_{2k}$ are uniquely determined from Condition (2.1) according to section I.

Thus, for elements of the matrix N we find

$$\frac{\partial \tilde{\nu}_i}{\partial \nu_j} = \sum_{\ell=1}^{2k} \frac{\partial \tilde{\nu}_i}{\partial t_\ell} \cdot \frac{\partial t_\ell}{\partial \nu_j}, \qquad (i, j = 2, 3, \dots m) \tag{2.13}$$

Let us consider Condition (2.1) in more detail. Under these conditions, (for the switching points) the inequality may have a maximum at two points corresponding to the boundary of the segment $[a, b]$, if $\Gamma(a) + \lambda > 0$ and $\Gamma(b) + \lambda > 0$.

It is apparent that then we may set

$$\frac{\partial a}{\partial \nu_j} = 0; \qquad \frac{\partial b}{\partial \nu_j} = 0; \qquad (j = 2, 3, \dots m) \tag{2.14}$$

These conditions may be readily interpreted geometrically. In actuality, let us set $\nu_2, \nu_3 \dots \nu_m ; \lambda$ are such that $\Gamma(\alpha)+\lambda>0$. Then the infinitely small deviation of the independent variable $\nu_2, \nu_3, \dots \nu_m$ does not shift the switching point α, i.e., $\frac{\Delta\alpha}{\Delta\nu_j}=0 \quad (j=2,3,\dots m)$ whereas the switching points lying within $[\alpha,\beta]$ are shifted to small values, due to a change in $\Gamma(t)$ and λ. In a similar way, we may explain the second condition (2.14). In the general case, different combinations may hold, such as when α and β are not included in the family of 2κ points, or only one of them is included in this set.

Thus, let us set /44

$$t_1, t_2, \dots t_n, \qquad 2\kappa-2\le n\le 2\kappa, \tag{2.15}$$

is a set of switching points which satisfy the equation $\Gamma(t_s)+\lambda=0$, $(s=1,2,\dots n)$.

(In unusual cases, we may have the equation $\Gamma(\alpha)+\lambda=0 \;\; \Gamma(\beta)+\lambda=0$, i.e., α and β may also be included in the set of n points). Taking into account condition (2.14) we find

$$\frac{\partial\tilde{\nu}_i}{\partial\nu_j}=\sum_{e=1}^{n}\frac{\partial\tilde{\nu}_i}{\partial t_e}\cdot\frac{\partial t_e}{\partial\nu_j}, \qquad (i,j=2,3,\dots m). \tag{2.16}$$

For the first factors in (2.16) (i is fixed), in view of (2.8) we have

$$\frac{\partial\tilde{\nu}_i}{\partial t_e}=\frac{\partial}{\partial t_e}\left(\frac{B_{1i}}{B_{11}}\right)=\frac{1}{B_{11}}\left(\frac{\partial B_{1i}}{\partial t_e}-\tilde{\nu}_i\,\frac{\partial B_{11}}{\partial t_e}\right) \tag{2.17}$$

The derivative $\frac{\partial B_{1i}}{\partial t_e}$ may be calculated as follows. The matrix relation is

$$BJ=E,$$

where E is the unit matrix which is differentiated with respect to t_e. Let us find

$$\frac{\partial B}{\partial t_\ell} = -B\frac{\partial J}{\partial t_\ell}B,$$

We may readily find that /45

$$\frac{\partial B_{1i}}{\partial t_\ell} = (-1)^{\ell+\ell_0+1}\cdot\gamma_1(t_\ell)\cdot\gamma_i(t_\ell) \tag{2.18}$$

where

$$\gamma_k(t_\ell) = \sum_{\tau=1}^{m} B_{k\tau}L_\tau(t_\ell), \quad (k=1,2,\ldots m;\ \ell=1,\ldots n), \tag{2.19}$$

$$\ell_0 = \begin{cases} 0, & \text{if} \quad \Gamma(a)+\lambda<0, \\ 1, & \text{if} \quad \Gamma(a)+\lambda>0. \end{cases} \tag{2.20}$$

Taking into account the above relationships, we find

$$\frac{\partial \tilde{\nu}_i}{\partial t_\ell} = (-1)^{\ell+\ell_0+1}\frac{\gamma_1(t_\ell)}{B_{11}}\left[\gamma_i(t_\ell)-\tilde{\nu}_i\gamma_1(t_\ell)\right], \quad (i=2,3,\ldots m;\ \ell=1,2,\ldots n). \tag{2.21}$$

The expressions for the second factors in (2.16) assume a somewhat more complex form.

Now let us first write all the conditions which are satisfied by the quantities $t_1, t_2, \ldots t_n;\ \nu_2, \nu_3, \ldots \nu_m$. We have

$$\begin{gathered} \Gamma(t_s)+\lambda = 0, \quad (s=1,2,\ldots n), \\ -t_1+t_2-\ldots(-1)^n t_n = c' \end{gathered} \tag{2.22}$$

where $c'=c_1$, if $t_1 \neq a, t_{2k} \neq b, c'=c_1-b$, if $t_1 \neq a$, $t_{2k}=b$, etc. The quantity c' does not interest us directly, since we can calculate only the derivatives $\frac{\partial t_\ell}{\partial \nu_j}$, $(\ell=1,\ldots n;\ j=2,3,\ldots m)$.

Excluding the quantity λ from (2.22), we arrive at the system of equations /48

$$\begin{cases} \Gamma(t_2)-\Gamma(t_1)=0 \\ \cdots\cdots\cdots\cdots \\ \Gamma(t_n)-\Gamma(t_1)=0 \\ -t_1+t_2-\ldots(-1)^n t_n=c' \end{cases} \tag{2.23}$$

Thus, in (2.28) we have n equations of n functions $t_1, t_2, \ldots t_n$, and of $(m-1)$ variables $\nu_2, \nu_3, \ldots \nu_m$. The partial derivatives $\frac{\partial t_\ell}{\partial \nu_j}$ $(\ell=1,2,\ldots n;$ $j=2,3,\ldots m)$ can in this case be calculated by the customary rules for derivatives of functions which are given implicitly.

Differentiating the system of equations (2.28) with respect to ν_j (j is fixed), we obtain n relationships

$$\begin{cases} \frac{\partial \Gamma_2}{\partial \nu_j}+\frac{\partial \Gamma_2}{\partial t_2}\cdot\frac{\partial t_2}{\partial \nu_j}-\frac{\partial \Gamma_1}{\partial \nu_j}-\frac{\partial \Gamma_1}{\partial t_1}\frac{\partial t_1}{\partial \nu_j}=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial \Gamma_n}{\partial \nu_j}+\frac{\partial \Gamma_n}{\partial t_n}\frac{\partial t_n}{\partial \nu_j}-\frac{\partial \Gamma_1}{\partial \nu_j}-\frac{\partial \Gamma_1}{\partial t_1}\frac{\partial t_1}{\partial \nu_j}=0, \\ -\frac{\partial t_1}{\partial \nu_j}+\frac{\partial t_2}{\partial \nu_j}-\frac{\partial t_3}{\partial \nu_j}+\ldots(-1)^n\frac{\partial t_n}{\partial \nu_j}=0, \end{cases} \tag{2.24}$$

where the notation $\Gamma_s \equiv \Gamma(t_s), (s=1,2,\ldots n)$ is introduced.

After calculating the derivatives of Γ_s, $(s=1,2,\ldots n)$ with respect to ν_j and t_z, $(z=1,2,\ldots n)$ we obtain a system whose matrix notation has the form

$$P y_j = \mathcal{F}_j, \tag{2.25}$$

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where P is the square matrix of order n:

$$P = \begin{pmatrix} -1 & +1 & -1 & \dots & (-1)^n \\ -p_1 & p_2 & 0 & \dots & 0 \\ -p_1 & 0 & p_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -p_1 & 0 & 0 & \dots & p_n \end{pmatrix}, \tag{2.26}$$

where

$$p_s = \sum_{z=1}^{m} \left\{ \nu_z \left[\frac{dL_z(t)}{dt} \right]_{t=t_s} \right\}, \quad (s=1,2,\dots n), \quad \nu_1 = 1;$$

and the columns $\mathcal{Y}_j$ and $\mathcal{F}_j$ have the form

$$\mathcal{Y}_j = \begin{pmatrix} \frac{\partial t_1}{\partial \nu_j} \\ \frac{\partial t_2}{\partial \nu_j} \\ \vdots \\ \frac{\partial t_n}{\partial \nu_j} \end{pmatrix}, \quad \mathcal{F}_j = \begin{pmatrix} 0 \\ L_j(t_1) - L_j(t_2) \\ \vdots \\ L_j(t_1) - L_j(t_n) \end{pmatrix}. \tag{2.27}$$

Assuming that the matrix P from (2.26) is not singular, we obtain

$$\mathcal{Y}_j = P^{-1} \mathcal{F}_j. \tag{2.28}$$

To find all the necessary derivatives $\frac{\partial t_\ell}{\partial \nu_j}$ $(\ell = 1,2,\dots n; \quad j=2,3,\dots m)$ /48
in the matrix Expression (2.28) we merely change the column $\mathcal{F}_j$, and the matrix P remains the same for all $j=2,3,\dots m$.

Let us consider one particular case when all the elements of the matrix $\mathcal{N}$ in (2.12) equal zero. Suppose there are only two points at which the direction is changed, where one of them (α or β) $t = t_0$ corresponds to an endpoint of the interval $[\alpha, \beta]$ and $\Gamma(t_0) + \lambda > 0$.

Then we have:

$$\frac{\partial \tilde{\nu}_i}{\partial \nu_j} = \frac{\partial \tilde{\nu}_i}{\partial t_0} \cdot \frac{\partial t_0}{\partial \nu_j} + \frac{\partial \tilde{\nu}_i}{\partial t_1} \cdot \frac{\partial t_1}{\partial \nu_j},$$
$$t_1 - t_0 = c'',$$

From (2.14) we obtain the desired result. Then the system of Equations (2.9) becomes:

$$\Delta \nu_i = -\xi_i, \qquad (i=2,3,\ldots m).$$

In all remaining cases, the elements of the matrix $\bar{\alpha} = \| \alpha_{ij} \|$ are calculated using the formulas given above.

The elements of the matrix $\mathcal{N}$ for the case of $M>1$ sets of measurements are similarly computed. We shall give the final formulas for these derivatives.

Suppose that M sets of points at which the direction is changed correspond to the Conditions (2.1'):

$$t_1^{(1)}, t_2^{(1)}, \ldots t_{n_1}^{(1)}; \;\ldots\ldots\ldots;\; t_1^{(M)}, t_2^{(M)}, \ldots t_{n_M}^{(M)};$$

where

$$2k_1 - 2 \leq n_1 \leq 2k_1; \;\ldots\ldots;\; 2k_M - 2 \leq n_M \leq 2k_M.$$ /49

These sets do not include the endpoints of the interval $[a, b]$ if they happen to be points at which the direction is changed, and satisfy the inequalities in (2.1').

The elements of the Matrix (2.12) in this case can be written in the following way:

$$\frac{\partial \tilde{\nu}_i}{\partial \nu_j} = \sum_{s=1}^{M} \sum_{\ell=1}^{n_s} \frac{\partial \tilde{\nu}_i}{\partial t_\ell^{(s)}} \cdot \frac{\partial t_\ell^{(s)}}{\partial \nu_j}, \quad (i,j = 2,3,\dots m) \tag{2.29}$$

The leading factors in (2.29) are

$$\frac{\partial \tilde{\nu}_i}{\partial t_\ell^{(s)}} = (-1)^{\ell + \ell_0^{(s)} + 1} \frac{\gamma_1^{(s)}(t_\ell)}{\sigma_s^2 \cdot B_H} [\gamma_i^{(s)}(t_\ell) - \tilde{\nu}_i \gamma_1^{(s)}(t_\ell)], \quad (2.21')$$

$$(i = 2,3,\dots m; \quad \ell = 1,2,\dots n_s; \quad s = 1,2,\dots M),$$

where

$$\gamma_k^{(s)}(t_\ell) = \sum_{z=1}^{m} \{B[V(t)]\}_{kz} \cdot L_z^{(s)}(t_\ell), \quad (k = 1,2,\dots m), \tag{2.21'}$$

$$\ell_0^{(s)} = \begin{cases} 0 & \text{when } \Gamma^{(s)}(a) + \lambda < 0, \\ \mathrm{I} & \text{when } \Gamma^{(s)}(a) + \lambda > 0. \end{cases}$$

The second factor in (2.29) can be obtained by solving the following system of matrix equations:

$$P^M y_j^M = \mathcal{F}_j^M, \quad (j = 2,3,\dots m),$$ /50

where

$$P^M = \begin{pmatrix} -c_0^{(1)} & c_0^{(1)} \dots & (-1)^{n_1} c_0^{(1)} & \dots & \dots -c_0^{(M)} & c_0^{(M)} \dots & (-1)^{n_M} c_0^{(M)} \\ -p_1^{(1)} & p_2^{(1)} \dots & 0 & \dots & \dots 0 & 0 \dots & 0 \\ \dots & \dots \dots & \dots & \dots & \dots \dots & \dots \dots & \dots \\ -p_1^{(1)} & 0 \dots & p_{n_1}^{(1)} & \dots & \dots 0 & 0 \dots & 0 \\ \dots & \dots \dots & \dots & \dots & \dots \dots & \dots \dots & \dots \\ \dots & \dots \dots & \dots & \dots & \dots \dots & \dots \dots & \dots \\ -p_1^{(1)} & 0 \dots & 0 & \dots & \dots p_1^{(M)} & 0 \dots & 0 \\ \dots & \dots \dots & \dots & \dots & \dots \dots & \dots \dots & \dots \\ -p_1^{(1)} & 0 \dots & 0 & \dots & \dots 0 & 0 \dots & p_{n_M}^{(M)} \end{pmatrix},$$

$$p_i^{(s)} = \left\{ \sum_{k=1}^{m} \nu_k L_k^{(s)}(t) \right\} \cdot \left\{ \sum_{z=1}^{m} \nu_z \frac{dL_z^{(s)}(t)}{dt} \bigg|_{t=t_i} \right\} \cdot \frac{1}{\sigma_s^2};$$

$$(i = 1,2,\dots n_s; \quad s = 1,2,\dots M),$$

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$$\mathcal{Y}_j^M = \begin{pmatrix} \dfrac{\partial t_1^{(1)}}{\partial \nu_j} \\ \dfrac{\partial t_2^{(1)}}{\partial \nu_j} \\ \vdots \\ \dfrac{\partial t_n^{(1)}}{\partial \nu_j} \\ \vdots \\ \vdots \\ \dfrac{\partial t_1^{(m)}}{\partial \nu_j} \\ \vdots \\ \dfrac{\partial t_{n_m}^{(m)}}{\partial \nu_j} \end{pmatrix}, \quad \mathcal{F}_j^M = \frac{1}{2} \begin{pmatrix} \vdots \quad 0 \\ \dfrac{\partial \Gamma_1^{(1)}}{\partial \nu_j} - \dfrac{\partial \Gamma_2^{(1)}}{\partial \nu_j} \\ \vdots \\ \dfrac{\partial \Gamma_1^{(1)}}{\partial \nu_j} - \dfrac{\partial \Gamma_n^{(1)}}{\partial \nu_j} \\ \vdots \\ \dfrac{\partial \Gamma_1^{(1)}}{\partial \nu_j} - \dfrac{\partial \Gamma_1^{(m)}}{\partial \nu_j} \\ \vdots \\ \dfrac{\partial \Gamma_1^{(1)}}{\partial \nu_j} - \dfrac{\partial \Gamma_{n_m}^{(m)}}{\partial \nu_j} \end{pmatrix},$$

$$\frac{\partial \Gamma_k^{(s)}}{\partial \nu_j} = \frac{\partial \Gamma^{(s)}(t_k)}{\partial \nu_j} = 2\left\{\sum_{i=1}^{m} \nu_i L_i^{(s)}(t_k^{(s)})\right\} L_j(t_k^{(s)}) \frac{1}{\sigma_s^2},$$

$$(k = 1, 2, \ldots n_s; \quad s = 1, 2, \ldots m).$$

It is clear that to determine $\mathcal{Y}_j^M$ the matrix j remains the same when the index P^M is varied; only the column $\mathcal{F}_j^M$ changes.

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4. Search for a Solution in the Case of Discontinuous Derivatives

Let us consider the process of solving the System (2.4) for a homogeneous set of measurements. By considering the matrix $\mathcal{N}$ [see (2.12)], it is easy to see that, if the number of points n corresponding to the switching times (2.15) is changed (in the descent toward a solution), then the derivatives appearing in the matrix $\mathcal{N}$ suffer a discontinuity. In this case, the relations used to form these derivatives show that the elements of the matrix $\mathcal{N}$ change discontinuously. This follows, for example, from the fact that a change in n implies a change in the order of the matrix P in (2.26), which means that the values of $\dfrac{\partial t_\ell}{\partial \nu_j}$, $(\ell = 1, 2, \ldots n; \ j = 2, 3, \ldots m)$, in terms of which the elements of the matrix N are directly expressed, also change discontinuously.

Discontinuities of the first kind in the elements of the matrix N mean that there are directions of descent along which a section of the unknown functional Φ in (2.5) has breaking points.

In this case, the functional Φ in the space ν represents a composite surface composed of $\rho > 1$ hypersurfaces for each of which the elements of the matrix N are continuous ($\rho = 1$, if in the descent process the number n remains the same, and a solution of the problem can be obtained using a modified method of steepest descent).

Due to the presence of discontinuities in the elements of the matrix N, the direction of steepest descent for the functional Φ does not generally correspond to the direction of the antigradient. In fact, the singular points of the line of cut of the hypersurface (points where the number n changes) /53
possess the property that their ε -neighborhoods do not determine a close direction of descent. (This fact is very important in the case when the lines of cut are nonconvex.) For continuous derivatives making up the matrix N, the ε -neighborhood of any point on the hypersurface defines close directions of descent.

The process of converging on a desired solution (for $\rho > 1$) can be achieved in the following way.

Suppose there is an initial approximation $\nu_2^{(0)}, \dots \nu_m^{(0)}$ to which there corresponds the number $n = n^{(0)}$. We take a step according to point 2; we obtain $n = n^{(1)}$. The basic comparison is that between $n^{(0)}$ and $n^{(1)}$. Then, if $n^{(0)} = n^{(1)}$ the problem proceeds according to the formulas in Section 2. However, after a certain number of steps (because $\rho > 1$) we obtain:

$$n^{(s)} \neq n^{(s+1)} \qquad (2.30)$$

This means that the point passed through one or several singular breaking points; here, the preceding hypersurface does not lead directly to a solution.

The Condition (2.30) is accompanied by one of the inequalities:

$$\Phi^{(s+1)} > \Phi^{(s)},$$
$$\Phi^{(s+1)} < \Phi^{(s)}.$$

If we arrive at the Inequality (2.32), the calculation is continued according to the standard scheme. In this case, the Condition (2.32) may occur at each step in the sequel. This case may correspond to a convex functional.

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In a general case, the Inequality (2.31) may hold simultaneously with the Condition (2.30).

Our next problem is to analyze the hypersurface cut in the direction of descent from the point s to the point $(s+1)$.

Above all, it is necessary to find all the singular points of this cut: τ_1, $\tau_2, \ldots \tau_\mu$. Then the points $\tau_1-\varepsilon$, $\tau_1+\varepsilon$, $\tau_2-\varepsilon$, $\tau_2+\varepsilon, \ldots \tau_\mu-\varepsilon$, $\tau_\mu+\varepsilon$, where ε is a small number $(\varepsilon > 0)$, are consecutively taken as the new initial approximations, and a step is taken which permits us to obtain a variation of the functional (2.5) resulting from a change in the initial approximation. The new initial approximation in subsequent calculations is taken to be the point corresponding to the maximum decrease of the functional. If in a given cut there are points which lead to a decrease of the functional during the step that follows, we then begin to analyze the hypersurface cut from the point $\Phi^{(s-1)}$ to the point $\Phi^{(s)}$.

The algorithm presented here was run many times on a computer, and proved its high efficiency.

III. Optimum Periods for Measuring the Radial Velocity to Determine the Orbit of an Artificial Satellite of Mars /55

1. Introduction of a Coordinate System; Isochronous Derivatives

Suppose that on the Earth, measurements of the radial velocity $\psi(t) = D$ are made to determine the orbit parameters of an artificial satellite of Mars.

Let us introduce a coordinate system. The origin O will be located at the center of mass of the planet Mars. The line of sight will be defined as the line connecting the centers of mass of Earth and Mars. It is clear that in absolute space the line of sight describes a ruled surface. Let the x axis be directed toward Earth, and suppose that at time $t = a$ it coincides with the line of sight. The plane xoy lies in the plane which is tangent to the ruled surface at time $t = a$, and the y axis is chosen in such a way that the positive angle from x toward y is measured counterclockwise when viewed from the north pole. The z axis completes the coordinate system in such a way as to make it right-handed.

It is assumed that the motion of an artificial satellite in its orbit around Mars is Keplerian and unperturbed. Suppose that this motion is described by the following parameters: r_π (pericenter distance), a (semimajor axis), τ (the time an artificial satellite of Mars passes through the pericenter), Ω (the longitude of the ascending mode, measured from the x axis), i (inclination of the orbit to the plane xoy), ω (angular distance of the pericenter from the node). /56

In the case of the orbit of an artificial satellite of Mars, the parameter r_π is a major characteristic (i.e., the variance of r_π is minimized).

In solving the problem, we have assumed a simplified model because of the large distance of the orbit of an artificial Mars satellite from Earth, and also since — to determine an optimal program of measurements — it is only necessary to know the isochronous derivatives.

We shall briefly state the principal simplifications.

Since the mean angular velocities of the revolution of Mars and Earth around the Sun are small, this means that the angular velocity of the displacement of the line of sight $\Omega_{\mathcal{E}}$ is also small, and within an interval of one or two months it may be considered constant in our model. If the interval of measurement does not exceed 24 hours, then the displacement of the line of sight may be generally neglected, i.e., we may set $\Omega_{\mathcal{E}} = 0$. Accounting for the angular dimensions of an artificial Mars satellite and the rotation of the Earth has very little effect on the isochronous derivatives.

Thus within a small measurement interval the function to be measured is in the form of the x component of the Mars-centric motion of the artificial Mars satellite.

In a general case $(\Omega_{\mathcal{E}} \neq 0)$, the unknown function to be measured becomes:

$$\psi(t) = -\dot{\tilde{x}}, \tag{3.1}$$

where

$$\dot{\tilde{x}} = \dot{x}\cos[\Omega_{\mathcal{E}}(t-a)] + \dot{y}\sin[\Omega_{\mathcal{E}}(t-a)], \tag{3.2}$$ /57

where $\dot{x}$, $\dot{y}$ are the velocity components of the Mars-centric unperturbed motion of an artificial Mars satellite [10]:

$$\dot{x} = \alpha\dot{\xi} + \alpha'\dot{\zeta}, \qquad \dot{y} = \beta\dot{\xi} + \beta'\dot{\zeta};$$
$$\alpha = \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i,$$
$$\alpha' = -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i, \tag{3.3}$$

(continued)

$$\beta = \cos\omega \sin\Omega + \sin\omega \cos\Omega \cos i,$$
$$\beta' = -\sin\omega \sin\Omega + \cos\omega \cos\Omega \cos i, \tag{3.3}$$

$$\dot{\xi} = -\sqrt{\frac{\mu}{p}} \sin\theta; \qquad \dot{\zeta} = \sqrt{\frac{\mu}{p}} (e + \cos\theta).$$

Here we use the notation: θ is the true anomaly,

$$r = \frac{p}{1 + e\cos\theta}, \quad p = \frac{a^2 - (a - r_\pi)^2}{a}; \quad e = 1 - \frac{r_\pi}{a}; \quad \mu = \kappa m,$$

(κ is the gravitational constant, m is the mass of Mars).

Equation (3.2) can be rewritten as

$$\dot{\tilde{x}} = \tilde{\alpha}\dot{\xi} + \tilde{\alpha}'\dot{\zeta}, \tag{3.4}$$

where $\tilde{\alpha}$ and $\tilde{\alpha}'$ can be obtained from α and α' after replacing Ω with $\tilde{\Omega} = \Omega - \Omega_6(t - a)$ in them.

It will be noted that, depending on the initial position of the planets, the value of Ω_6 may differ as to its sign.

We introduce the notation: /58

$$L_1 = \frac{\partial\Psi}{\partial r_\pi}, \quad L_2 = \frac{\partial\Psi}{\partial a}, \quad L_3 = \frac{\partial\Psi}{\partial \tau}, \quad L_4 = \frac{\partial\Psi}{\partial \Omega}, \quad L_5 = \frac{\partial\Psi}{\partial i}, \quad L_6 = \frac{\partial\Psi}{\partial \omega}, \tag{3.5}$$

then we obtain the following for the measured function (3.1) [10]:

$$L_1 = \frac{1}{p}\sqrt{\frac{\mu}{p}}(\tilde{\alpha}' W_1 - \tilde{\alpha} W_2),$$
$$L_2 = (1 - e)\frac{1}{p}\sqrt{\frac{\mu}{p}}(\tilde{\alpha}' W_3 - \tilde{\alpha} W_4), \tag{3.6}$$

(continued)

$$L_3 = -\mu(\tilde{\alpha}' V_6 + \tilde{\alpha} V_7),$$
$$L_4 = -\sqrt{\frac{\mu}{p}}[\tilde{\beta} V_0 - \tilde{\beta}'(e + V_1)],$$
$$L_5 = \sqrt{\frac{\mu}{p}} \sin i \sin\tilde{\Omega}[\sin\omega V_0 - \cos\omega(e + V_1)],$$
$$L_6 = \sqrt{\frac{\mu}{p}}[\tilde{\alpha}' V_0 + \tilde{\alpha}(e + V_1)], \quad (3.6)$$

where $\tilde{\beta}$ and $\tilde{\beta}'$ can be obtained from β and β' [see (3.3)] after replacing Ω by $\tilde{\Omega}$. The remaining unknown quantities in (3.6) are defined by the ten linearly independent functions of the variable θ:

$$\begin{array}{ll} V_0 = \sin\theta, & V_1 = \cos\theta, \\ V_2 = \sin\theta\cos\theta, & V_3 = \cos^2\theta, \\ V_4 = \sin\theta\cos^2\theta, & V_5 = \cos^3\theta, \\ V_6 = \dfrac{\sin\theta}{r^2}, & V_7 = \dfrac{\cos\theta}{r^2}, \\ V_8 = \dfrac{\sin\theta}{r^2}(t - \tau), & V_9 = \dfrac{\cos\theta}{r^2}(t - \tau); \end{array} \quad (3.7)$$

Among those, W_1, W_2, W_3, W_4 have the form: /59

$$W_1 = eV_5 + 2V_3 - 1,$$
$$W_2 = eV_0 + 2V_2 + eV_4,$$
$$W_3 = -W_1 + \frac{1+e}{2}(e + V_1 - 3\sqrt{\mu p}\, V_8),$$
$$W_4 = -W_2 + \frac{1+e}{2}(V_0 + 3\sqrt{\mu p}\, V_9). \quad (3.8)$$

2. Certain Characteristic Features of Determining the Optimum Program of Measurements in the Problem Under Consideration

First of all, it should be noted that a determination of an optimum measurement program is possible only when the number of parameters is such that the latter can be determined with a given set of measurements.

One of the most important practical cases in the determination of an orbit of an artificial Mars satellite is the case where it is required to determine the orbit within a short period of time — say, within less than 24 hours. In this case, in estimating the accuracy of the prediction one may neglect the orbital motion of the planets, i.e., one may set $\Omega_\ell = 0$.

It is obvious that in this case it is impossible to estimate the accuracy of all six parameters of the orbit of an artificial Mars satellite, since to a given measured function $\psi = -\dot{x}$ there corresponds a family of orbits obtained by a rotation about the x axis. Mathematically, this indeterminacy implies that for isochronous derivatives taken with respect to the angular parameters /60 there exists a linear relationship which in this case can be written as:

$$L_6(t) = \hat{a} L_Y(t) + \hat{b} L_5(t), \tag{3.9}$$

where $\hat{a} = \cos i$, $\hat{b} = -ctg\eta \sin i$, and $\sin\eta \neq 0$. If $\sin\eta = 0$, then $L_5(t) \equiv 0$, $t \in [a, b]$, $\sin i \neq 0$.

The Condition (3.9) implies that $\det J = 0$. Therefore, to estimate the accuracy of the six parameters of the artificial Mars satellite orbit using a short measurement interval, we must have another set of measurements or a measurement of $\dot{D}$ with the a priori information about one of the angular parameters.

The orbit of an artificial Mars satellite using only the measurements of $\dot{D}$ can only be determined within a relatively long measurement interval when a displacement of the line of sight makes it possible to fix the spatial orientation of the orbit.

Suppose that $\Omega_\ell = 0$. Consider a case in which the optimum program of measurements (the variance τ_T is minimized) is to be determined in the following two problems: 1) the optimum program of measurements of $\dot{D}$ is found with five determinable parameters $Q = \{\tau_T, a, \tau, \omega, i\}$; 2) the optimum program is found with six determinable parameters $Q = \{\tau_T, a, \tau, \omega, i, \eta\}$

with a priori information about the parameter η. It is easy to see that the optimum programs of measurements coincide in these two problems.

In fact, denoting the Matrices (2.2) for these two cases by $\mathcal{J}$ and $\mathcal{J}'$, /61
we obtain $\nu_2 = \nu_2'$, $\nu_3 = \nu_3'$, $\nu_4 = \nu_4'$, $\nu_5 = \nu_5'$, $\nu_6' = 0$ considering (2.3). This proves the above assertion.

Now let us consider the orbits of an artificial Mars satellite whose parameters can only poorly be determined when measuring $\dot{D}$. For $\Omega_\varepsilon = 0$ these will primarily be orbits that lie in the coordinate planes:

$$
\begin{array}{llll}
\mathrm{I} & i = \frac{\pi}{2}, & \cos\eta = 0 & (yoz) \\
\mathrm{II} & i = \frac{\pi}{2}, & \sin\eta = 0 & (xoz) \\
\mathrm{III} & i = 0, & \cdot & (xoy)
\end{array}
\qquad (3.10)
$$

For the first group of orbits yoz we have, according to (3.6): $L_1(t) = L_2(t) = L_3(t) = L_6(t) = \Psi(t) \equiv 0$ (within the interval of measurement, where $\Omega_\varepsilon = 0$). This is the most unfavorable case (the plane of the orbit is perpendicular to the direction toward the Earth), and the orbit is practically undeterminable. For orbits lying in the plane xoz, we have $L_4(t) \equiv L_5(t) \equiv 0$. For Group III we have: $L_5(t) \simeq 0$ for $i \simeq 0$; the parameter η loses its meaning.

For orbits with the parameters $i \neq \frac{\pi}{2}$, $\sin\eta = 0$, we have $L_5(t) \equiv 0$.

For $\Omega_\varepsilon \neq 0$ there is only one singular case: these are orbits which lie in the plane xoy.

Below we give the results of calculations, given for two important cases: $\Omega_\varepsilon = 0$. $(m=5)$ and $\Omega_\varepsilon \neq 0$ $(m=6)$.

3. The Case of a Fixed Line of Sight /62

Let $\Omega_\ell = 0$. We shall give the results of calculations for the optimum program of measurements $\dot{D}$ (the variance of τ_π is being minimized) for the five parameters: τ_π, a, τ, ω, i. The parameter Ω is considered known.

In this problem, the majority of the integrals appearing in the matrix $\mathcal{J}$ in (2.2) may be taken in a closed form; only three integrals in the matrix $\mathcal{J}$ are not expressible as quadratures. An approximate evaluation of these integrals may be done using Gauss' method [13].

The following values were used for the parameters of the orbit of an artificial Mars satellite: $\tau_\pi = 4944.34$ km, $a = 20085.10$ km (the period of revolution of an artificial Mars satellite is $T = 24^h.000$), $\tau = 0$, $\Omega = 30^\circ$, $i = 45^\circ$, $\omega = 60^\circ$. In addition, we took: $\mu = 42850$ km^3/sec^2.

The allowable interval of measurements was taken to be $[0, T]$.

The variational problem in this case becomes nonsingular; its solution was obtained by the method presented in Section 4,II (the value of $\frac{C_1}{C_0}$ was varied).

We shall briefly consider the process of convergence to the solution of the System (2.4) for $\frac{C_1}{C_0} = 6912^3$. (See Table I.) The initial approximation was taken to be the following: $\nu_2^{(0)} = \nu_3^{(0)} = \nu_4^{(0)} = \nu_5^{(0)} = 0$, $[\Gamma^{(0)}(t) = L_1^2(t)]$, which corresponds to $n^{(0)} = 6$ and $\Phi^{(0)} = 1.278$. Four measurement periods in the region of the pericenter correspond to this approximation: two near the point $t = \tau = 0$ and two near the point $t = T$. The following step gives $\Phi^{(1)} < \Phi^{(0)}$ and then $\Phi^{(2)} = 0.041 < \Phi^{(1)} = 0.093$. /64

After this, in the following step we obtain $\Phi^{(3)} = 6.95 >> \Phi^{(2)}$. To a cut in the direction from the point $s = 2$ to the point $s = 3$ there correspond singular points whose ε-neighborhoods result in an increase of the functional.

TABLE I. THE CHARACTERISTIC VALUES IN SOLVING THE SYSTEM OF TRANSCENDENTAL EQUATIONS

$$t \in [0, T], \quad \frac{C_1}{C_0} = 6912^{s}.$$

s	ν_2	$-\nu_3$	$-\nu_4 \cdot 10^3$	$-\nu_5 \cdot 10^2$	n	$\Phi^{(s)}$
0	0.00000	0.00000	0.00000	0.000000	6	I.278
I	0.I6452	0.62588	0.23646	0.06702	7	0.093
2*	0.I8030	0.660I4	0.26956	0.07893	8	0.037
3*	0.22I43	0.80246	0.40035	0.II475	II	0.035
4*	0.20I43	0.73352	0.37489	0.I070I	9	0.0I7
5*	0.2I553	0.78642	0.39852	0.II334	II	$6.6.10^{-3}$
6*	0.206I7	0.75260	0.38545	0.I0950	II	$5.9.10^{-3}$
7*	0.2I477	0.78357	0.39805	0.II323	II	$4.4.10^{-3}$
8*	0.2I064	0.76873	0.39202	0.III44	II	$7.0.10^{-6}$
9*	0.2I086	0.7695I	0.3924I	0.III55	II	$7.1.10^{-7}$
I0*	0.2I08I	0.76937	0.39242	0.III55	II	$5.4.10^{-9}$
II*	0.2I08I	0.76935	0.3924I	0.III55	II	$3.2.10^{-11}$
I4*	0.2I0808	0.769356	0.3924II	0.III550	II	$2.9.10^{-13}$
		values of $\tilde{\nu}_j$ for 14 steps:				
	0.2I0808	0.769356	0.3924II	0.III550		

Because of this, the point $s=3$ is thrown away, and then we consider a cut from the point $s=1$ to $s=2$, according to the algorithm described in Section 4,II. To make this clearer, the cut is shown in Figure 1. In it, the point $\tau=0$ corresponds to $\Phi^{(1)}=0.093$, the point $\tau=1$ corresponds to $\Phi^{(2)}=0.041$. It can be seen that the line of cut of the desired hypersurface is nonconvex and belongs to the three hypersurfaces for which $n=7,8,9$. The singular points are denoted by τ_1, τ_2. Furthermore, according to the

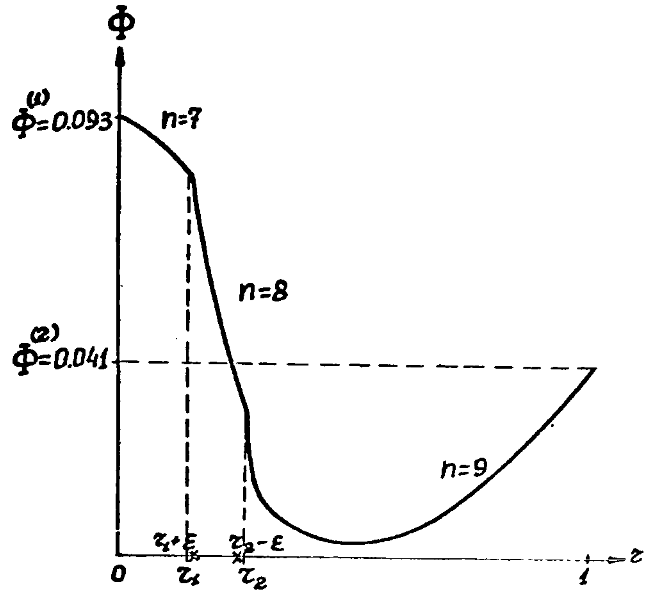

Figure 1. The cut of the hypersurface $\Phi(\nu_2, \nu_3, \nu_4, \nu_5)$ in the direction of descent from the point $\Phi^{(1)}$ (Table 1).

algorithm presented,we study the process of convergence of four points: $\tau_1-\varepsilon$, $\tau_1+\varepsilon$, $\tau_2-\varepsilon$, $\tau_2+\varepsilon, (\varepsilon>0)$, taken consecutively as the second approximation. Only the points $\tau_1+\varepsilon$ and $\tau_2-\varepsilon$ which belong to one hypersurface for which $n=8$, lead in the following step to a decrease of the functional, and the point $\tau_2-\varepsilon$ gives the greatest decrease of the functional. This point was taken as the new initial approximation (the step $s=2^*$), which leads to a solution of the problem for $s=14^*$. Thus, after the point $s=2^*$ the functional becomes smaller with every step, and as a result, the computation was continued according to the above scheme involving a modified method of steepest descent. The point $\tau_1+\varepsilon$ leads to the same extremum solution, except that the process of convergence here is considerably slower.

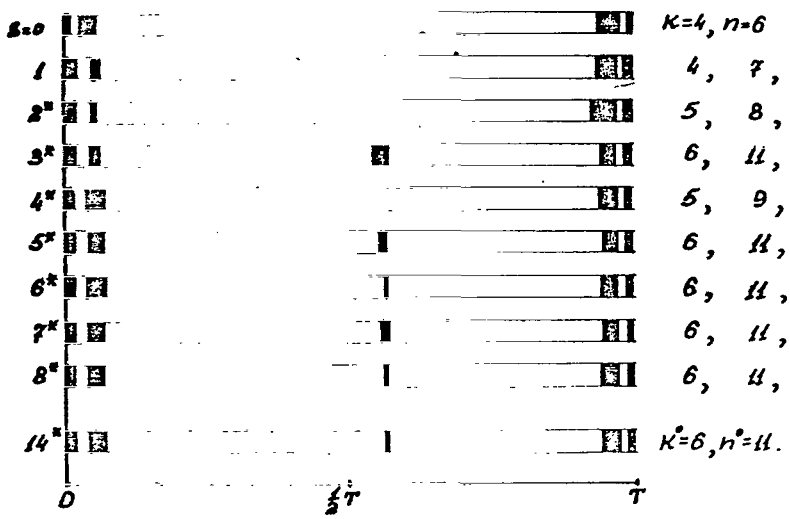

Figure 1a. The variation of the measurement periods (cross-hatched regions) in solving systems of transcendental equations (S-N step; see also Table I).

Table I reproduces the solution process for the point $\tau_2 - \mathcal{E}$. In the 14th step we encounter a difference between ν_j and $\tilde{\dot{\nu}}_j$ (j = 2,3,4,5) in the seventh significant digits which corresponds to the accuracy with which this problem is solved. It will be noted that the following errors were assumed: /67

$\mathcal{E}_K$ =10^{-2} sec (the error involved in calculating the time of direction change),

$\mathcal{E}_{c_1}$ =10^{-1} sec (the error involved in satisfying the integral restriction). The variation of measurement periods in solving the problem under consideration is shown in Figure 1a.

Six periods correspond to the extremum solution, of which only one lies in the region of the apocenter (its length is $\sim 10^m$). This interval, as was found, occurs in the solution only in the case when the parameter i is being determined. In fact, the optimum programs of measurements for the same value of $\frac{C_1}{C_0}$ for m = 2,3,4 do not have the measurement interval in the apocenter. All these programs are close to each other, and contain measurement intervals only in the vicinity of the pericenter.

Here and below the values of $\Delta Q_j = \sqrt{B_{jj}}$, $(j=2,3,\dots m)$ are given for $\sigma = 1$ m/sec and $C_0 = \frac{1}{10}$ measurements/sec (one measurement per 10 sec).

Figure 2 gives the optimum measurement periods $\dot{D}$ for various values of $\frac{C_1}{C_0}$ ($\Delta \tau_T$ is everywhere minimized). The orbit of an artificial Mars satellite is given to scale (the figure is planar). The planet Mars is taken to be a sphere of radius 3400 km.

The measurement periods (cross-hatched regions) are given with respect to the true anomaly (on the orbits), and with respect to the time (spectra) for the values $\frac{C_1}{C_0} = 1^h, 2^h, 4^h, 12^h$. It can be seen in Figure 2 that an increase in $\frac{C_1}{C_0}$ leads to the maximum widening of the measurement interval in the region /69 of the apocenter. Thus, for $\frac{C_1}{C_0} = 2^h$ the length of this interval is $\sim 0^h 17$; for $\frac{C_1}{C_0} = 4^h$ it is $\sim 1^h.59$; for $\frac{C_1}{C_0} = 12^h$, it is $\sim 7^h.81$. For $\frac{C_1}{C_0} = 1^h$ we have only five measurement intervals corresponding to the region of the pericenter of an artificial Mars satellite. A lowering of $\frac{C_1}{C_0}$ to 20^m has shown that these periods have a tendency to converge to five points. This result agrees with the theorem of Elving-Yershov, since here $m=5$. However, an increase in $\frac{C_1}{C_0}$ implies that $K^0 = 6$. (K^0 is the number of optimum measurement periods.) This circumstance is related to the presence of a restriction on the density of measurements when it becomes inconvenient to increase the duration of the periods (-points) corresponding to the theorem of Elving-Yershov. It is more effective to make measurements in the region of the apocenter. For $\frac{C_1}{C_0} \sim 5^h$ the length of the interval in the region of the apocenter will coincide with the cumulative duration of all intervals in the pericenter.

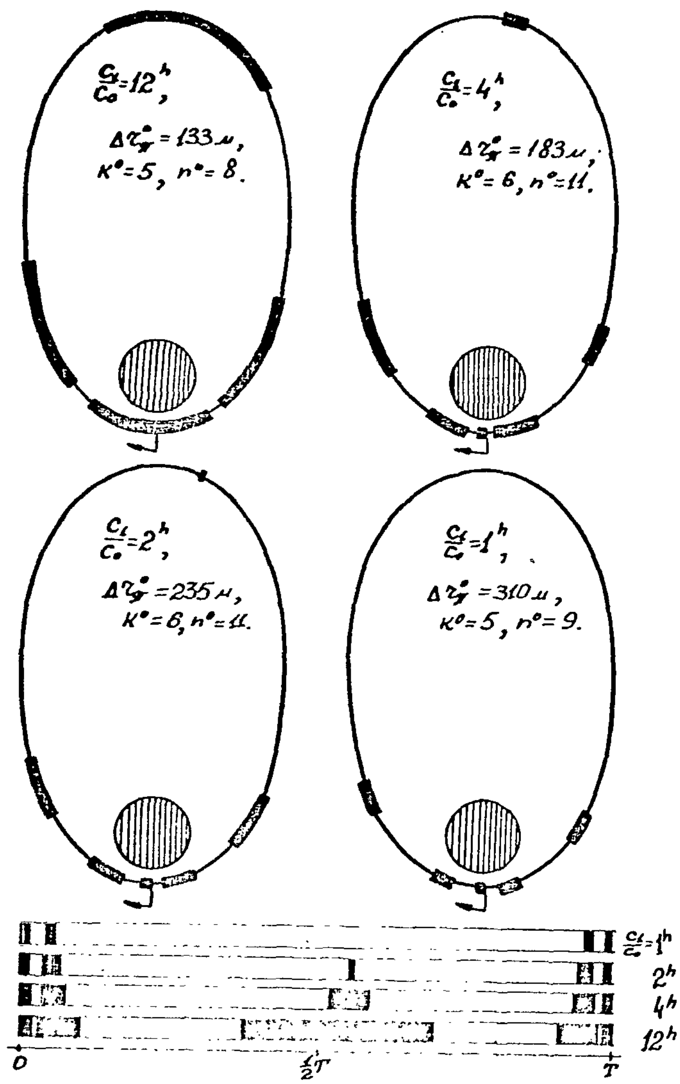

Figure 2. The optimum measurement periods of $\dot{D}$ during one revolution of an artificial Mars satellite (m = 5). The measurement periods are plotted against the time (spectra), and against the true anomaly (on the orbits).

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TABLE II. AN EXTREMUM SOLUTION IN THE SEGMENT $[0,T]$ ($u^0(t)=c_0$ IN THE INTERVALS $[t_{2l}, t_{2l-1}]$ $l=1,2,\ldots k^0$; IN THE REMAINING INTERVALS $[0,T]-u^0(t)=0$) AND THE CORRESPONDING CHARACTERISTIC VALUES ($\Omega_6=0, m=5$) .

$\frac{c_1}{c_0}$	$\frac{T}{24}=1^h$	$\frac{T}{12}=2^h$	$\frac{T}{6}=4^h$	$\frac{T}{2}=12^h$
k^0, n^0	5; 9	6; II	6; II	5; 8
V_2^0	0.I987I	0.20959	0.I82II	0.III99
$-V_3^0$	0.73457	0.76447	0.65695	0.39762
$-V_4^0 \cdot 10^3$	0.38372	0.389I3	0.3I65I	0.I3849
$-V_5^0 \cdot 10^2$	0.I08I2	0.II064	0.09038	0.03944
t_1^0 (hr)	0.000	0.000	0.000	0.000
t_2^0	0.044	0.055	0.025	0.485
t_3^0	0.208	0.I78	0.I27	0.72I
t_4^0	0.344	0.4I0	0.43I	2.444
t_5^0	I.078	0.983	0.904	8.97I
t_6^0	I.429	I.668	I.807	I6.779
t_7^0	22.862	I3.428	I2.529	2I.848
t_8^0	23.I82	I3.600	I4.I20	23.445
t_9^0	23.743	22.636	22.454	23.6I3
t_{10}^0	23.892	23.254	23.3I8	24.000
t_{11}^0		23.682	23.659	
t_{12}^0		23.920	23.972	

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$\frac{c_1}{c_0}$	1^h	2^h	4^h	12^h
$\Phi^0=\sum_{i=2}^{5}(\tilde{V}_i^0-V_i^0)$	2.5.IO^{-II}	I.0.IO^{-I2}	2.7.IO^{-I2}	$\sim$IO^{-I6}
$\Delta \tau_T^0$ (M)	3I0	235	I83	I33
Δa (M)	95	77	63	37
$\Delta \tau$ (sec)	0.39	0.3I	0.25	0.I6
$\Delta \omega'$	0.64	0.52	0.4I	0.23
$\Delta i'$	I.78	I.46	I.I7	0.64

For $\frac{C}{C_0} \doteq 12^h$ the interval in the region of the apocenter is the longest.

The numerical values of the characteristic magnitudes in this problem are listed in Table II.

4. Taking Into Consideration Displacement in the Line of Sight /72

An estimate of accuracy for six parameters of the orbit of an artificial Mars satellite for measuring the radial velocity is possible only when considering displacements of the line of sight.

Thus, let $\Omega_6 \neq 0$. We shall take $\Omega_6 = 10^{-7}$ rads/sec ~ 0.5 deg., mean time ($\Delta\tau_T$ is minimized).

If one solves the problem for the interval $[0, T]$, by virtue of the smallness of Ω_6 the matrix J will have a weak basis.

Let us consider the interval $[0, 2T]$ and $Q = \{\tau_T, a, \tau, \Omega, i, \omega\}$ (numerical values of the parameters were given in Section 3).

Within this interval, one can obtain the solution of the extremum problem in question. It will be noted that the maximum deviation of the line of sight in this measurement interval amounts to one degree.

A determination of the optimum measurement program within the interval $[0, 2T]$ will be stable. Some results of these calculations are given in Figure 3, in Table 3, 4.

In Figure 3 the measurement periods are plotted against the true anomaly by means of solid-line segments on two branches of a twisting spiral (the arrow shows the direction of motion of an artificial Mars satellite), and against the time in the form of spectra. The diagram of the orbit of an artificial Mars satellite is planar (the diagram is to scale).

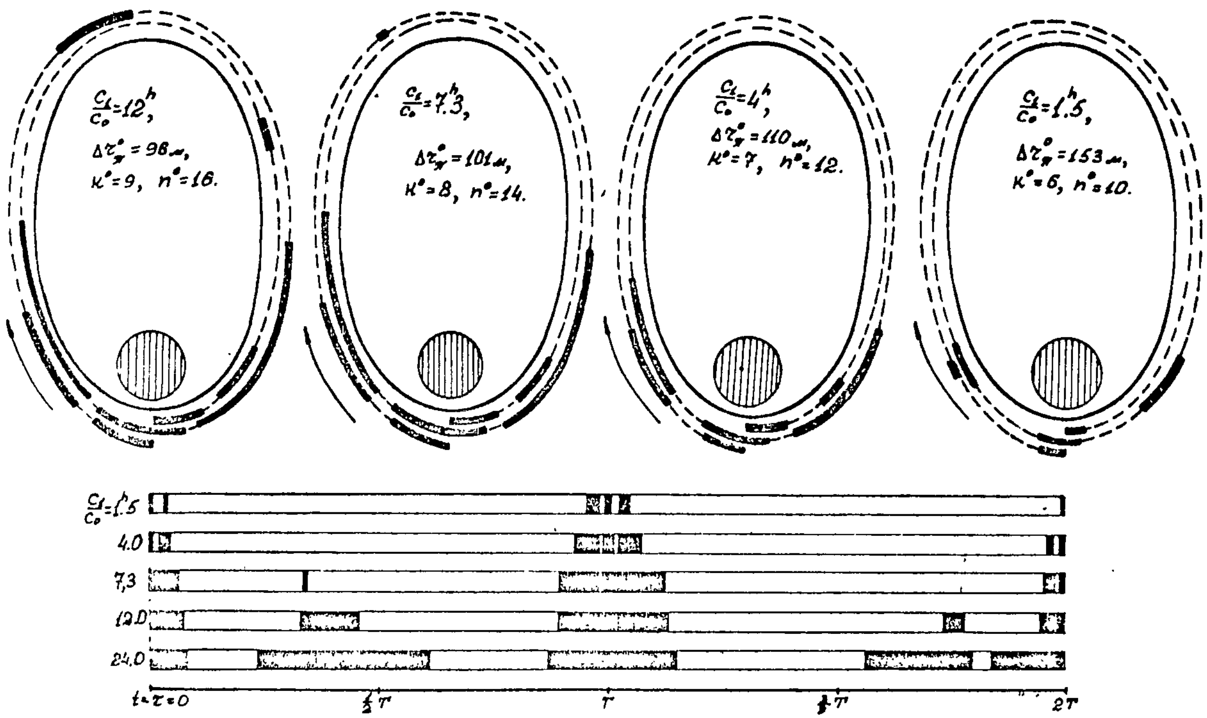

Figure 3. Optimum measurement periods of $\dot{D}$ for two revolutions of an artificial Mars satellite (m=6). Measurement periods are shown with solid-line segments on two branches of a twisting spiral (with respect to the true anomaly), and in the form of spectra (with respect to the time).

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The extremum solution in the interval $[0, 2T]$ contains from six to nine optimum measurement periods depending on $\frac{C_1}{C_0}$.

For $\frac{C_1}{C_0} \to 0$ we have six points (this result agrees with the theorem of Elving-Yershov). All these points correspond to the region of the pericenter of an artificial Mars satellite, where two points lie at the origin ($t = \tau = 0$), three lie near the time $t = T$, and one at the time $t = 2T$. In this case, the distribution of measurements is extremely nonuniform. As an example, let us consider the case $\frac{C_1}{C_0} = 1^h.5$. The first two periods (the region $t = \tau = 0$) have a total length of about 15^m. The succeeding three periods (near $t = T$) have a total length $\sim 1^h 9^h$, and finally, the period which ends at $t = 2T$ has the length of only 6^m. The last measurement period has a short duration as compared with the remaining five periods. /74

For all values of $\frac{C_1}{C_0}$ the number of measurements during the first revolution is greater than during the second one. Thus, for $\frac{C_1}{C_0} = 1^h5$; 4^h; 6^h; $7^h.3$; 12^h; 24^h, respectively, the total measurement interval during the first revolution of an artificial Mars satellite exceeds that during the second revolution: $0^h.14$; $0^h.30$; $0^h.14$; $0^h.08$; $2^h.10$; $3^h.26$.

The fundamental evolution of the extremum solution from the limiting value of the extremum solution corresponding to the parameter $\frac{C_1}{C_0} \to 0$ is clearly apparent.

Thus, for $\frac{C_1}{C_0} = 24^h$ the optimum measurement periods of greatest length correspond to the region of the apocenter of the first and second revolutions of an artificial Mars satellite; the total number of periods is equal to nine. A decrease in $\frac{C_1}{C_0}$ results at first only in a considerable shortening of these intervals (they later vanish). For $\frac{C_1}{C_0} = 7^h.3$ the number of the extremum periods is equal to eight, and their number during the first revolution is greater (for this value of $\frac{C_1}{C_0}$ the interval near the apocenter during the first revolution of an artificial Mars satellite is important). A further decrease in $\frac{C_1}{C_0}$ implies that there only remain the optimum measurement periods

TABLE III. THE EXTREMUM SOLUTION WITHIN THE INTERVAL $[0.2T]$ $(\Omega_6 \neq 0, m=6)$. THE OPTIMUM CONTROL $u^{\circ}(t) = C_o$ IN THE SEGMENTS $[t_{2l}, t_{2l-1}]$ $(l = 1,2,\ldots K^{\circ})$ AND $u^{\circ}(t) = 0$ IN THE REMAINING INTERVALS $[0.2T]$

t_j^{o} (hours) \ $\frac{C_1}{C_o}$	$1^h.5 = \frac{2T}{32}$ $(K^o = 6)$	$4^h.0 = \frac{2T}{12}$ $(K^o = 7)$	$7^h.3 \approx \frac{2T}{6.6}$ $(K^o = 8)$	$12^h.0 = \frac{2T}{4}$ $(K^o = 9)$	$24^h.0 = T$ $(K^o = 9)$
$j = 1$	0.000	0.000	0.000	0.000	0.000
2	0.II7	0.222	0.294	0.34I	0.387
3	0.806	0.63I	0.49I	0.5I4	0.5I0
4	0.932	I.I58	I.488	I.732	2.062
5	23.000	22.395	8.087	7.839	5.864
6	23.504	23.637	8.I54	I0.897	I4.725
7	23.928	23.845	2I.575	2I.483	2I.I04
8	24.I48	24.283	23.698	23.686	23.66I
9	24.722	24.6I8	23.797	23.774	23.729
10	25.I53	25.846	24.366	24.379	24.4I4
11	47.898	47.246	24.529	24.520	24.520
12	48.000	47.366	27.006	27.2II	27.52I
13	-	47.777	47.023	4I.829	37.800
14	-	48.000	47.484	42.679	43.272
15	-	-	47.688	46.803	46.407
16	-	-	48.000	47.508	47.537
17	-	-	-	47.67I	47.645
18	-	-	-	48.000	48.000

in the region of the pericenter of an artificial Mars satellite. There are seven of them within $[0, 2T]$ (for example, for $\frac{C_1}{C_0} = 4^h$). Finally, for $\frac{C_1}{C_0} = 1^h.5$ we have only six measurement periods. /76

The most stable with respect to a change of the parameter $\frac{C_1}{C_0}$ were those measurement periods which include the moment of passage through the pericenter (using the time scale, these will be the three points: $t=\tau=0$; $t=T$; $t=2T$). Thus, for $\frac{C_1}{C_0} = 2^h$ the duration of these periods is equal to $0^h.15$ $(t=\tau=0)$, $0^h.28$ $(t=T)$, $0^h.13$ $(t=2T)$, and for $\frac{C_1}{C_0} = 24^h$ we have $0^h.39$, $0^h.78$, $0^h.35$, respectively. (It should be noted that for $\frac{C_1}{C_0} = 24^h$ these three periods have minimum lengths as compared with the remaining six, see Figure 3.) Thus, when $\frac{C_1}{C_0}$ is made 12 times smaller, the periods in question decrease in length by less than a factor of three, whereas the remaining periods either generally disappear or decrease by a greater factor.

This example shows the considerable difference which may take place between solving a problem with a restriction on the rate of measurements and solving a problem without this restriction (the theorem of Elfving-Yershov).

Now it is necessary to talk about the change $\Delta \alpha_j$ ($j = 1, 2 \ldots 6$). The quantity $\Delta \tau_\pi$ varies little from $\frac{C_1}{C_0}$ in the case of the characteristic parameter. Thus, for the parameter $\frac{C_1}{C_0}$ equal to $1^h.5$ and $24^h.0$, we have $\Delta \tau_\pi$ equal to 153 m and 90 m, respectively. For the same values of $\frac{C_1}{C_0}$ the errors in the angular parameters Ω, i, ω are $25'$, $44'$, $36'$ and $7'$, $12'$, $10'$, respectively. The error in τ in this case decreases by a factor of 4, and the error in the period decreases more than three times. /78

The solution of the problem in this case ($\Omega_\varepsilon \neq 0$) was obtained by descent with respect to the parameter $\frac{C_1}{C_0}$. Initially, an extremum solution was found using the algorithm presented above for $\frac{C_1}{C_0} = 12^h.0$. The initial approximation was taken as: $\nu_2^{(0)} = \nu_3^{(0)} = \ldots = \nu_6^{(0)} = 0$.

TABLE IV. THE CHARACTERISTIC VALUES CORRESPONDING TO THE EXTREMUM SOLUTION WITHIN THE INTERVAL $[0, 2T]$ (SEE TABLE III)

The values of ΔQ_j, $(j=1,2,\ldots 6)$ were obtained with $\Delta\dot{D}(=\dot{\sigma})=1$ m/sec, $C_0=\frac{1}{10}$ measurements/sec.

$\frac{C_1}{C_0}$	$1^h.5$	$4^h.0$	$7^h.3$	$12^h.0$	$24^h.0$
$-\nu_2^0 \cdot 10^2$	0.I376	0.92638	I.3367	0.76437	0.67653
$-\nu_3^0 \cdot 10^2$	3.302	I.9635	0.23780	2.76400	0.66287
$+\nu_4^0 \cdot 10^2$	0.I356	0.50802	0.7I959	0.57443	0.45555
$-\nu_5^0 \cdot 10^2$	0.2597	0.90625	I.2755	I.0I523	0.79595
$-\nu_6^0 \cdot 10^2$	0.2003	0.7228I	I.020I	0.8I364	0.64I93
$\Phi=\sum_{i=2}^{6}(\tilde{\nu}_i^0-\nu_i^0)^3$	3.I0^{-II}	I0^{-II}	8.I0^{-I4}	5.I0^{-I4}	3.I0^{-I6}
$\Delta \tau_{\pi}^0$ (m)	I53	II0	I0I	96	90
Δa (M)	58	30	22	20	I7
$\Delta \tau$ (sec)	0.47	0.20	0.I4	0.I3	0.I2
$\Delta \Omega'$	25.I	I5.5	II.2	9.0	7.0
$\Delta i'$	44.I	27.2	I9.7	I5.8	I2.3
$\Delta \omega'$	35.5	2I.8	I5.8	I2.7	9.9

The extremum solution obtained was used to obtain the solution for smaller values of $\frac{C_1}{C_0}$ by means of a descent with respect to the parameter $\frac{C_1}{C_0}$. (However, also here it was necessary to cut the hypersurfaces along the direction of descent toward the minimum of the functional).

Such a modification was used here, since an arbitrary initial approximation for small values of $\frac{C_1}{C_0}$ may result in an unstable matrix B — for example, if to the initial approximation there correspond two measurement periods during the first revolution of an artificial Mars satellite.

It must be noted that, when the parameter $\frac{C_1}{C_0}$ becomes larger, the $\mathcal{B}$ matrix becomes more stable. This is, in particular, indicated by the fact that in the case of the extremum solution the quantity (2.5) changes from 10^{-11} to 10^{-16} when $\frac{C_1}{C_0}$ increases from $1^h.5$ to $24^h.0$. The stability of the matrix B is here very important, since the stability of the entire solution process depends on it by virtue of the fact that during each step the inverse of the matrix $\mathcal{J}$ must be found.

To increase the stability of the computation in this problem, the elements /79 of the matrix $\mathcal{J}$ were obtained using integration by means of the Gauss method [13]. The points where a change of direction took place were determined in each step with an accuracy to within 0.1 - 0.01 seconds.

The calculations that were performed have shown that the proposed algorithm for finding the optimum program of measurements is highly efficient.

In conclusion, the author wishes to take advantage of this opportunity and express his deep appreciation to T. M. Eneyev for having formulated the problem, his constant attention given to this work, and a great deal of help obtained from him.

REFERENCES

1. Akim, E. L. and T. M. Eneyev. Kosmicheskiye issledovaniya, Vol. 1, No. 1, 1963.

2. Elfving, G. Ann. Math. Stat., Vol. 23, 1952, p. 255.

3. Yershov, V. G. Kosmicheskiye issledovaniya, Vol. 8, No. 1, 1970, p. 8.

4. Faddeyev, D. K. and V. N. Faddeyeva. Vychislitel'nyye metody lineynoy algebry (Computational Methods of Linear Algebra), Fizmatgiz (State Publishing House of Physics and Mathematics Literature), Moscow-Leningrad, Vol. 3, 1960.

5. Fikhtengol'ts, G. M. Kurs differentsial'nogo i integral'nogo ischisleniya (A Course of Differential and Integral Calculus), Fizmatgiz, Moscow-Leningrad, Vol. 3, 1960.

6. Okhotsimskiy, D. Ye. Prikladnaya matematika i mekhanika, Vol. 10, No. 2, 1946, p. 251.

7. Okhotsimskiy, D. Ye and T. M. Eneyev. Uspekhi fizicheskikh nauk, Vol. 63, No. 1a, 1957, p. 5.

8. Kachmazh, S. and G. Shteyngauz. Teoriya ortogonal'nykh ryadov (Theory of Orthogonal Series), Fizmatgiz, M., 1958.

9. Rudakov, V. M. Kosmicheskiye issledovaniya, Vol. 7, No. 3, 1969, p. 352.

10. Duboshin, G. N. Nebesnaya mekhanika. Osnovnyye zadachi i metody (Celestial Mechanics. Fundamental Problems and Methods), Fizmatgiz, Moscow, 1963.

11. Eneyev, T. M., A. K. Platonov and R. K. Kazakova. Collection: Iskusstvennyye sputniki Zemli (Artificial Earth Satellites), No. 4, 1960, p. 43.

12. Optimization Methods with Applications to Space Flight Mechanics, edited by J. Deytman. Translated from the English, "Nauka" Press, Moscow, 1965.

13. Kronrod, A. S. Uzly i vesa kvadraturnykh formul (Nodes and Weights in Quadrature Equations), "Nauka" Press, Moscow, 1964.

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